ENUMERATION INVOLVING SUMS OVER COMPOSITIONS

by

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The undersigned certify that they have read and recommend to the Faculty of Graduate Studies for acceptance, a thesis entitled "ENUMERATION INVOLVING SUMS OVER COMPOSITIONS", submitted by DAVID A. KLARNER in partial fulfilment of the requirements for the degree of Doctor of Philosophy.

ABSTRACT

In this thesis we have presented a theory for sums having the form

$$(1) \qquad b(n) = \sum f(a_1,a_2)\, f(a_2,a_3) \ldots f(a_{i-1},a_i)\, g(a_i) ,$$

where the sum extends over all compositions $(a_1,a_2,\ldots,a_i)$ of n into an unrestricted number of positive parts. The numbers $\{f(m,n)\}$ and $\{g(n)\}$ are given by generating functions $F(x,y)$ and $G(x)$ respectively, and we seek a relationship between these functions and the generating function $B(x,y) = \sum_{n,a} b(a,n)\, y^a x^n$, where $b(a,n)$ denotes the partial sum obtained from (1) by summing over just those compositions of n with $a_1 = a$. We show in Chapter I that

$$(2) \qquad B(x,y) = G(xy) + \frac{1}{2\pi i} \int_c F(xy,\tfrac{1}{s})\, B(x,s)\, \frac{ds}{s} ,$$

where c is a contour in the s plane which includes the singularities of $F(xy,\frac{1}{s})/s$, but excludes those of $B(x,s)$. The relation (2) is a special case of the Fredholm integral equation which can be solved by standard methods described in Chapter I .

Chapter II begins with a concise definition of an n-omino, and a description of the work done on the cell growth problem. Eden [1] proved that the following special case of (1) gives a lower bound for

the number of n-ominoes:

$$(3) \qquad b(n) = \sum (a_1+a_2-1)(a_2+a_3-1) \ldots (a_{i-1}+a_i-1) .$$

Using the methods described in Chapter I, we find the generating function of $\{b(n)\}$ as defined by (3); also, we find this generating function by an elementary method in order to show, in one instance at least, the advantage of treating the Fredholm equation.

In Chapter III we prove there are more than $(3.6)^n$ n-ominoes for all sufficiently large n . To do this we prove that

$$(4) \qquad f(a_1,a_2,\ldots,a_i) \geq f(a_1,a_2)\ f(a_2,a_3) \ldots f(a_{i-1},a_i) ,$$

where $(a_1,a_2,\ldots,a_i)$ denotes any composition of n , and $f(p,q,\ldots)$ denotes the number of n-ominoes with p cells in the first row, q cells in the second row, and so on. Next, we find the generating function of $\{f(m,n)\}$, and use the Fredholm equation to find a lower bound for

$$(5) \qquad b(n) = \sum f(a_1,a_2)\ f(a_2,a_3) \ldots f(a_{i-1},a_i) ,$$

which is in turn a lower bound for the number of n-ominoes.

Read [17] stated and proved a general combinatorial theorem which he used to find the number of n-ominoes with 1,2,3,4, or 5 rows of cells. In Chapter IV we solve Read's problem by means of the Fredholm equation. Also, we indicate a method for finding the number

of n-ominoes with k rows using linear difference equations.

Harary, Prins, and Tutte [6] used Pólya's theorem to enumerate planted plane trees; also, they gave a one-one correspondence between the planted plane trees with $n+2$ nodes and the (tri-valent) planted plane trees with $2n+2$ nodes in which each node has valence either 1 or 3. In Chapter V we show that the methods of Chapter I apply to these enumeration problems as well, and go on to enumerate special classes of planted plane trees, for example, planted planes trees having nodes with valance either 1 or k. Furthermore, we give a very simple correspondence between the planted plane trees and the tri-valent planted plane trees by matching these trees with the elements of a certain set of binary codes.

We discuss the connection between the Fredholm equation and the generating functions for certain arithmetic functions in Chapter VI. Sums extended over the divisors of n may be thought of as sums extended over just those compositions of n in which all of the parts are the same. Also, by defining

$$(6)\qquad f(m,n) = \begin{cases} f^*(m,n), & \text{if } m \leq n \\ 0, & \text{otherwise} \end{cases},$$

the index of summation in (1) is actually restricted to the partitions $(a_1 \leq a_2 \leq \ldots \leq a_i)$ of n, since all other compositions of n only contribute 0 to the sum. Using this idea, we are able to establish a link between the Fredholm equation and the theory of partitions.

Chapter VII contains still another application of Pólya's fundamental theorem, this time to find the number of non-isomorphic generalized graphs with n nodes. A <u>generalized graph</u> is a pair (N_n, N), where $N_n = \{1,2,\ldots,n\}$ is the set of <u>nodes</u>, and N is some subset of the k-tuples $N_n \times N_n \times \ldots \times N_n$ called the <u>edge set</u> of the graph.

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CHAPTER I

GENERATING FUNCTIONS

Our objective in this chapter will be to develop a theory for treating certain sums which arise in a variety of combinatorial problems. The notation and accompanying definitions introduced at this time will be used in subsequent chapters without further explanation; of course a symbol defined here will have various interpretations later on, depending on the context in which it is used.

Suppose $\{f(m,n) : m, n = 1, 2, \ldots\}$ and $\{g(n) : n = 1, 2, \ldots\}$ are given sets of numbers and consider the numbers $b(n)$ defined by

$$(1) \qquad b(n) = \sum f(a_1,a_2)\, f(a_2,a_3) \ldots f(a_{i-1},a_i)\, g(a_i) \;,$$

where the sum extends over all compositions $(a_1,a_2,\ldots,a_i)$ of n into an unrestricted number of positive parts. We make the convention that $g(n)$ is the contribution to the sum when the number of parts of the composition is one.

The symbol $b_k^j(a,n)$, used with all or only some of the suffixes, will denote the partial sum obtained from (1) when the index of summation has been restricted to those compositions of n which have exactly k

parts, no part greater than j, and the first part equal to a; if a suffix is dropped, the corresponding restriction on the index of summation is dropped as well. This definition implies certain obvious relationships between the numbers $b_k^j(a,n)$ defined. For example, we have

$$(2) \qquad b(n) = \sum_{a=1}^{n} b(a,n) = \sum_{k=1}^{n} b_k(n) \quad ,$$

$$(3) \qquad b^j(n) = \sum_{a=1}^{n} b^j(a,n) = \sum_{k=1}^{n} b_k^j(n) \quad ,$$

$$(4) \qquad b(a,n) = \sum_{k=1}^{n} b_k(a,n), \quad b_k(n) = \sum_{a=1}^{n} b_k(a,n) \quad .$$

Less trivial perhaps are the following relationships which play a central role in later work: for $a < n$,

$$(5) \qquad b(a,n) = \sum f(a,a_1)\, f(a_1,a_2) \ldots f(a_{i-1},a_i)\, g(a_i)$$

$$= \sum_{\nu} f(a,\nu)\, b(\nu,n-a) \quad ,$$

$$(6) \qquad b_{k+1}(a,n) = \sum_{\nu} f(a,\nu)\, b_k(\nu,n-a) \; , \quad k = 1, 2, \ldots \quad .$$

The index of summation in the first sum in (5) is extended over all compositions $(a_1,a_2,\ldots,a_i)$ of $n-a$ into an unrestricted number of

positive parts; the second equality follows on summing over compositions of $n-a$ with $a_1 = \nu$ for $\nu = 1, 2, \ldots, n-a$. The proof of (6) is similar.

Now we turn our attention to finding the relationships between the generating functions of the sequences being studied. Thus, we define

$$(7) \qquad F(x,y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} f(m,n)\, x^m y^n , \quad G(x) = \sum_{n=1}^{\infty} g(n)\, x^n ,$$

$$B(x,y) = \sum_{n=1}^{\infty} \sum_{a=1}^{n} b(a,n)\, y^a x^n ,$$

so that $F(x,y)$ is an analytic function of x for fixed y and of y for fixed x in neighborhoods of $x = 0$ and $y = 0$ respectively, and $G(x)$ is an analytic function in a neighborhood of $x = 0$. For fixed natural numbers j and k we write $B^j(x,y)$ and $B_k(x,y)$ for the generating functions of $\{b^j(a,n)\}$ and $\{b_k(a,n)\}$ respectively. For example, using this notation, the relations in (2), (3) and (4) imply

$$(8) \qquad B(x,1) = \sum_{n=1}^{\infty} b(n)\, x^n ,$$

$$(9) \qquad B(x,y) = \sum_{k=1}^{\infty} B_k(x,y) .$$

All of the B functions are analytic if F and G are analytic; to show this it is sufficient to show that

$$(10) \qquad \sum_{n=1}^{\infty} \sum |f(a_1,a_2)||f(a_2,a_3)| \dots |f(a_{i-1},a_i)||g(a_i)| \, x^n$$

converges for sufficiently small $|x|$, where the inner sum extends over all compositions of n. The proof follows from the fact that series such as in (7) converge if, and only if, there exist positive constants α, β and γ such that $|f(m,n)| < \alpha^m\beta^n$ and $|g(n)| < \gamma^n$; assuming this we have for a given composition $(a_1,a_2,\dots,a_i)$ of n,

$$(11) \qquad |f(a_1,a_2)||f(a_2,a_3)| \dots |f(a_{i-1},a_i)||f(a_i)| < (\alpha\beta\gamma)^n \ .$$

There are 2^{n-1} compositions of n, so the coefficient of x^n in (10) is less than $(2\alpha\beta\gamma)^n$; hence, the series converges for $|x| < 1/(2\alpha\beta\gamma)$.

It is clear that the B functions are uniquely defined by the coefficients $b_k^j(a,n)$ once F and G are specified; however, there are other ways to construct these functions. We begin by finding a relationship between F, G and the sequence $\{B_k\}$.

By definition

$$(12) \qquad b_1(a,n) = \begin{cases} g(n), & \text{if } a = n \\ 0 \ , & \text{otherwise} \end{cases} ,$$

in terms of generating functions this becomes,

(13) $$B_1(x,y) = \sum_{n=1}^{\infty} g(n)\, x^n y^n = G(xy) \quad .$$

Next, multiplying through (6) with $y^a x^n$ and summing over a and n we obtain

(14) $$B_{k+1}(x,y) = \sum_{a,n} \sum_{\nu} f(a,\nu)\, b_k(\nu,n-a)\, y^a x^n \quad .$$

It is easy to verify by substituting the power series involved that the sum in the right member of (14) is the residue with respect to s of $F(xy,\frac{1}{s})\, B_k(x,s)/s$; hence, for $k = 1, 2, \ldots$,

(15) $$B_{k+1}(x,y) = \frac{1}{2\pi i} \int_c F(xy,\tfrac{1}{s})\, B_k(x,s)\, \frac{ds}{s} \quad ,$$

where c is a contour in the s plane which includes the singularities of $F(xy,\frac{1}{s})/s$, but excludes those of $B_k(x,s)$. In practice the singularities in the s plane of $F(xy,\frac{1}{s})/s$ may be known, in which case the integration in (15) may be carried out to obtain a recurrence relation satisfied by the sequence $\{B_k\}$. If we are able to surmise the form of B_k in general after calculating $B_1, B_2, \ldots$, this recurrence relation can be used to prove the conjectured formula by induction.

Summing on k in (15) we obtain

(16) $$\sum_{k=1}^{\infty} B_{k+1}(x,y) = \frac{1}{2\pi i} \int_c F(xy,\tfrac{1}{s}) \sum_{k=1}^{\infty} B_k(x,s)\, \frac{ds}{s} \quad ;$$

adding $B_1(x,y) = G(xy)$ to each side of the equality in (16) we obtain

(17) $$B(x,y) = G(xy) + \frac{1}{2\pi i} \int_C F(xy,\tfrac{1}{s})\, B(x,s)\, \frac{ds}{s} \quad .$$

It is also possible to find (17) directly from (5); here the proof involves verifying that

(18) $$\frac{1}{2\pi i} \int_C F(xy,\tfrac{1}{s})\, B(x,s)\, \frac{ds}{s} = \sum_{n=2}^{\infty} \sum_{a=1}^{n-1} \sum_{\nu=1}^{n-a} f(a,\nu)\, b(\nu,n-a)\, y^a x^n \quad ,$$

but this simply amounts to finding the coefficient of s^{-1} in the Laurent expansion of $F(xy,\frac{1}{s})\, B(x,s)/s$.

Just as in (15), the integral equation in (17) becomes a functional equation satisfied by $B(x,y)$ when the singularities of $F(xy,\frac{1}{s})/s$ are known; sometimes it will be possible to use this relationship to solve for $B(x,y)$ explicitly.

Now we are ready to consider the functions $B^j(x,y)$. First we note that $b^j(a,n)$ would correspond to the number $b(a,n)$ which would result if

(19) $$F_j(x,y) = \sum_{m=1}^{j} \sum_{n=1}^{j} f(m,n)\, x^m y^n \quad \text{and} \quad G_j(x) = \sum_{n=1}^{j} g(n)\, x^n$$

were used in place of F and G to define the sets $\{f(m,n)\}$ and $\{g(n)\}$. This observation along with (17) implies that

(20) $$B^j(x,y) = G_j(xy) + \frac{1}{2\pi i} \int_C F_j(xy,\tfrac{1}{s})\, B^j(x,s)\, \frac{ds}{s} \quad ,$$

where c includes the origin (since $F(xy,\frac{1}{s})/s$ is simply a polynomial in $1/s$) but excludes the singularities of $B^j(x,s)$. Substituting the representation of $F_j(xy,\frac{1}{s})$ given by (19) into (20) we obtain after integrating

$$(21)\qquad B^j(x,y) = \sum_{m=1}^{j} \left\{ g(m) + \sum_{n=1}^{j} f(m,n) \frac{\partial^n B^j}{n!} \right\} x^m y^m \quad , .$$

where $\partial^n B^j$ denotes the n^{th} partial derivative with respect to s of $B^j(x,s)$ at $s = 0$. But we also have

$$(22)\qquad B^j(x,y) = \sum_{m=1}^{j} \frac{\partial^m B^j}{m!} y^m \quad ,$$

so that equating coefficients of y^p in (21) and (22) gives the system of equations

$$(23)\qquad \frac{\partial^p B^j}{p!} = g(p)\, x^p + \sum_{n=1}^{j} f(p,n) \frac{\partial^n B^j}{n!} x^p \quad ,$$

for $p = 1, 2, \ldots, j$. The system in (23) consists of j equations, linear with respect to the functions $\partial^p B^j/p!$, so Cramer's rule may be applied to solve for these functions explicitly in terms of x and the numbers $g(n)$ and $f(m,n)$, $m,n = 1, 2, \ldots, j$. In fact, we have

$$(24)\qquad \frac{\partial^p B^j}{p!} = \Delta_{jp}(x)/\Delta_j(x) \quad ,$$

$$(25)\qquad \Delta_j(x) = \det\,[a_{rs}] \ , \quad \Delta_{jp}(x) = \det\,[b^p_{rs}] \quad ,$$

where $[a_{rs}]$ is defined by

$$(26) \qquad a_{rs} = \begin{cases} f(r,r)\, x^r - 1\,, & \text{if } r = s \\ f(r,s)\, x^r\,, & \text{if } r \neq s \end{cases}$$

and $[b^p_{rs}]$ is obtained from $[a_{rs}]$ by replacing the p^{th} column of $[a_{rs}]$ with the column vector $[-g(1)x, -g(2)x^2, \ldots, -g(j)x^j]$. Substituting the expressions for $\partial^p B^j/p!$ given by (24) into (22) gives

$$(27) \qquad B^j(x,y) = \frac{1}{\Delta_j(x)} \sum_{m=1}^{j} \Delta_{jm}(x)\, y^m \quad ,$$

so that evidently $B^j(x,y)$ is a rational function.

Part of our interest in the functions $b^j(a,n)$ is evinced by the fact that $b^j(a,n) = b(a,n)$ for $n \leq j$, and hence the implication that $B^j(x,y)$ tends uniformly to $B(x,y)$ in a suitable domain $D(x,y)$; this follows from the fact that

$$(28) \qquad B(x,y) - B^j(x,y) = \sum_{n=j+1}^{\infty} \sum_{a=1}^{n} \{b(a,n) - b^j(a,n)\}\, y^a x^n \quad .$$

In some of the problems we study later, we will have $0 \leq b^j(a,n) \leq b(a,n)$ with $\{b^j(a,n) : n = 1, 2, \ldots\}$ an increasing sequence for each j. For example, this will be the case when $f(m,n), g(n) \geq 1$, for $m,n = 1, 2, \ldots$.

If P and Q are polynomials such that Q/P is irreducible and generates an increasing sequence $\{a_n\}$, then $\lim_{n \to \infty} (a_n)^{1/n}$ exists and is the largest real root of $P(1/x) = 0$; a proof of this may be found in Salem [21]. Thus, if $B^j(x,1) = \Sigma \Delta_{jp}(x)/\Delta_j(x)$ generates an increasing sequence $\{b^j(n) : n = 1, 2, \ldots\}$ with $0 \leq b^j(n) \leq b(n)$, then

$$(29) \qquad (b(n))^{1/n} \geq (b^j(n))^{1/n} > \theta_j - \epsilon \quad ,$$

for all sufficiently large n, where θ_j is the largest real root of $\Delta_j(1/x) = 0$. If furthermore, $\{b^j(n) : j = 1, 2, \ldots\}$ is an increasing sequence for each n (this will also be the case if $f(m,n), g(n) \geq 1$, $m,n = 1, 2, \ldots$) then $\theta_1 \leq \theta_2 \leq \ldots$ is bounded by $\lim_{n \to \infty} (b(n))^{1/n} = \emptyset$ so that $\{\theta_j\}$ tends to a limit $\theta \leq \emptyset$. Thus, for every $\epsilon > 0$ we have

$$(30) \qquad (b(n))^{1/n} > \theta - \epsilon$$

for all sufficiently large n. So far we have been unable to discover whether $\theta = \emptyset$ under these assumptions.

The relation in (17) is a special case of an important integral equation known as the Fredholm equation; Whittaker and Watson [24] give the following definition. Let $\omega(z)$ be a continuous function and suppose $K(z,s)$ is a real function of z and s such that either (i) it is

a continuous function of both variables in the range $a \le s, \; z \le b$, or (ii) it is a continuous function of both variables in the range $a \le s \le z \le b$ and $K(z,s) = 0$ when $s > z$. Now

$$(31) \qquad \phi(z) = \omega(z) + \int_a^b K(z,s)\, \phi(s)\, ds$$

is Fredholm's equation; $K(z,s)$ is called the kernel. To see that the equation in (17) has this form, we suppose c is defined by $s(t)$ for $0 \le t \le 1$, regard x as a fixed number, write $y = s(z)$, and convert the contour integral into a line integral to obtain

$$(32) \qquad B(x,s(z)) = G(xs(z)) + \frac{1}{2\pi i} \int_0^1 F\left(xs(z),\frac{1}{s(t)}\right) B(x,s(t))\, \frac{ds(t)}{s(t)} \;.$$

This equation is less general than the Fredholm equation in (31) principally in that F and G are assumed to be analytic; consequently, the theory of the equation is considerably simpler in the case we consider. For example, the existence of a unique solution is guaranteed since the definition of $b(a,n)$ in (5) and the relationship between the generating series in (17) are equivalent statements.

The expression for $B(x,y)$ given by (9) corresponds to the solution of (32) given by the method of successive approximations and is called the Neumann series while the approximations of $B(x,y)$ given by the functions $B^j(x,y)$ correspond to the solution given by Fredholm's method. An excellent exposition of the theory of this integral equation as well as a description of the work done since Fredholm first gave a

complete theory for them in 1900, may be found in Riesz and Sz.-Nagy [18].

The connection between the Fredholm equation and convolutions of the kind given in (5) seems not to have been observed. As will be seen in subsequent chapters, certain arithmetical functions as well as combinational functions satisfy convolutions of this type.

CHAPTER II

CELL GROWTH PROBLEMS

The square lattice in the plane is defined by Hilbert [10] as the set of all points of the plane whose Cartesian coordinates are integers. A <u>cell</u> of the square lattice is a point set consisting of the boundary and interior points of a unit square having its vertices at lattice points. An <u>n-omino</u> is a union of n cells which is connected and has no finite cut set.

The set of all n-ominoes, R_n, is an infinite set for each n; however, we are interested in the elements of two finite sets of equivalence classes, S_n and T_n, which are defined on the elements of R_n as follows: Two elements of R_n belong to the same equivalence class (i) in S_n, or (ii) in T_n, if one can be transformed into the other by (i) a translation, or (ii) by a translation, rotation, and reflection of the plane. An element of R_n is in <u>standard position</u> if it is above the x-axis with a cell at the origin and all cells in the first row are to the right of the y-axis.

There is exactly one element in each equivalence class in S_n which is in standard position, while an equivalence class in T_n may contain as many as eight elements all in standard position. Thus,

if $s(n)$ and $t(n)$ denote the number of elements in S_n and T_n, then

(1) $$\frac{1}{8}\, s(n) \leq t(n) \leq s(n) \quad .$$

Harary [4] has listed the cell growth problem as an unsolved problem in the enumeration of graphs. Stated in the terms we have just defined, Harary's formulation of the cell growth problem is to find $t^*(n)$, the number of equivalence classes of T_n which contain simply connected n-ominoes. If an n-omino is not simply connected, in the sense that it has "holes", then it is said to be multiply connected. Harary [5] later reported that a computer had been programmed to find $t(n)$ for $n \leq 12$, and he listed these values. Evidently this work was carried out independently by Stein, Walden, and Williamson at the Los Alamos Scientific Laboratories and by Lander and Parkin at the Aerospace Corporation. This and more detailed information about the counts were communicated to the author by Mr. Parkin.

Read [17] calls representative elements of the equivalence classes of S_n and T_n fixed and free animals with n cells respectively; also, he gave a method for finding the number of equivalence classes of S_n which contain n-ominoes in standard position having cells in exactly k rows above the x-axis. He calculated these numbers for $k \leq 5$ and $n \leq 10$ and used these results to find $t(n)$ and $t^*(n)$ for $n \leq 10$; an error in his calculations involving $t(10)$ was discovered by Parkin and subsequently corrected by Read. The known

values of $t(n)$ and $t^*(n)$ are as follows:

	n	1	2	3	4	5	6	7	8	9	10	11	12
(2)	$t(n)$	1	1	2	5	12	35	108	369	1285	4655	17073	63600
	$t^*(n)$	1	1	2	5	12	35	107	363	1248	4271	———	———

Eden [1] seems to have been the first person to give upper and lower bounds for $t(n)$; his bounds are

(3) $$(3.14)^n < t(n) < 4^n \quad ,$$

for sufficiently large n ; his argument for the upper bound is in doubt.

We will show that $\alpha = \lim_{n \to \infty} (s(n))^{1/n}$ exists, so that from (1) we can conclude that $\lim_{n \to \infty} (t(n))^{1/n}$ exists and is equal to α ; to do this, we will require a lemma due to Fekete (see Pólya and Szego [15], page 171, for similar results).

<u>Lemma 1</u>: If $\{U_n\}$ is a sequence of natural numbers such that $\{(U_n)^{1/n}\}$ is bounded and $U_m U_n \leq U_{m+n}$, then $\lim_{n \to \infty} (U_n)^{1/n}$ exists.

To show that $\{s(n)\}$ satisfies the conditions of Lemma 1 we prove two more lemmas; in the proof of Lemma 3 we indicate the method Eden used to establish the upper bound given in (3).

<u>Lemma 2</u>: $s(m)\, s(n) \leq s(m+n)$, $m,n = 1, 2, \dots$.

Proof: Let X and Y be representative elements from equivalence classes in S_m and S_n respectively, such that the lower edge of the first cell in the bottom row of Y is joined to the upper edge of the last cell in the top row of X . The (m+n)-omino just described is a representative element of an equivalence class in S_{m+n} . The existence of this one-one correspondence between $S_m \times S_n$ and a subset of S_{m+n} imples the lemma.

Lemma 3: $s(n) < (27/4)^n$.

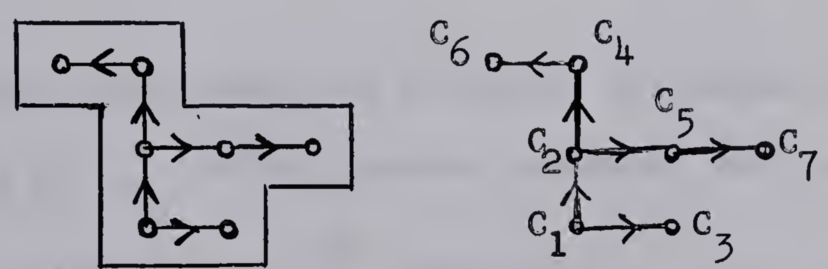
Proof: Following Eden, we assign a unique sequence of binary digits to each element X of S_n which will be denoted by $W(X) = \{(\alpha_1\beta_1)(\alpha_2\beta_2\gamma_2) \ldots (\alpha_n\beta_n\gamma_n)\}$. To do this, we assume X is in standard position and draw a labelled, directed tree over the cells of X ; the nodes of the tree will be the centers of the cells of X and the directed edges will be drawn between some of the nodes in connected cells. The center of the cell located at the origin is given the label C_1 , and directed edges are drawn to the centers of the cells connected to it; these nodes are called $C_2, \ldots,$ proceeding clockwise from the y-axis around C_1 . Now we describe a process that must be carried out at the nodes $C_2, C_3, \ldots, C_n,$ the nodes being taken in this order. Suppose the process has been carried out at $C_1, C_2, \ldots, C_{i-1},$ and that a directed edge has been drawn from C_j to C_i . Beginning with the side of C_i to the right of C_j and proceeding clockwise around C_i we draw a directed edge to any cell connected to C_i which has

not already been labelled, giving the labels C_k, C_{k+1}, ..., to the new nodes, where k is the smallest index not used previously. Now α_i, β_i, and γ_i are assigned the binary digits 1 or 0 if an edge going from C_i cuts the first, second, or third side encountered, proceeding clockwise from C_j around C_i.

The units which appear as digits in $W(X)$ correspond to the edges of the tree drawn over the cells of X; thus, the binary digits of $W(X)$ must sum to $n-1$, since a tree with n nodes has exactly $n-1$ edges. The number of binary sequences of length $3n-1$ that contain exactly $n-1$ ones is $\binom{3n-1}{n-1}$; since different elements of S_n give rise to different binary sequences, it follows that

$$s(n) \leq \binom{3n-1}{n-1} < \left(\frac{27}{4}\right)^n . \tag{4}$$

This completes the proof of Lemma 3; in Figure 1 we show a 7-omino in standard position with its labelled tree and corresponding binary sequence.



(11)(011)(000)(100)(010)(000)(000)

Figure 1.

Combining Lemmas 1, 2, and 3 we have the following result.

<u>Theorem 1</u>: $\lim\limits_{n \to \infty} (s(n))^{1/n} = \alpha$ exists, and $\alpha < 27/4$.

If there is a directed edge from C_i to C_j , and from C_j to C_k , then the choice of the binary digits in $(\alpha_i \beta_i \gamma_i)$ and $(\alpha_j \beta_j \gamma_j)$ usually imposes restraints on the choice of $(\alpha_k \beta_k \gamma_k)$. Using this idea, Eden claims to have shown that fewer than 4^n of the binary sequences of length $3n-1$ have digits which satisfy these restraints.

If an n-omino in standard position has exactly a_j cells in the j^{th} row above the x-axis, for $j = 1, 2, \ldots, i$, we say the equivalence class it represents in S_n belongs to the set $F_{a_1 a_2 \ldots a_i}$. Eden observed that if $f(a_1, a_2, \ldots, a_i)$ denotes the number of elements in $F_{a_1 a_2 \ldots a_i}$, then

(5) $$f(a_1, a_2, \ldots, a_i) \geq (a_1 + a_2 - 1)(a_2 + a_3 - 1) \ldots (a_{i-1} + a_i - 1) \quad .$$

The product in the right member of (5) gives the number of equivalence classes in $F_{a_1 a_2 \ldots a_i}$ which contain n-ominoes that have a connected strip of a_j cells in the j^{th} row, for $j = 1, 2, \ldots, i$, when the n-omino is in standard position. This follows from the fact that a strip of r cells can be joined above a strip of s cells in $r+s-1$ ways. Since there is a set $F_{a_1 a_2 \ldots a_i}$ corresponding to every composition of n , we have

Combining Lemmas 3.3 and 3.4 we have the following result.

Theorem 3.1 $\lim_{n \to \infty} (\phi(n))^{1/n}$ [illegible]

It should be [illegible] [illegible] [illegible] [illegible] σ_1 [illegible] $(\sigma_1\sigma_2\ldots)$ [illegible] $(\sigma_1\sigma_2)$ [illegible] $(\sigma_1\sigma_2)$. After this [illegible] [illegible] [illegible] [illegible] [illegible] connection.

[illegible] $\phi^{(2)}$ [illegible] [illegible] σ_n [illegible] [illegible] $\sigma(\sigma_1\sigma_2 \ldots \sigma_n)$ [illegible] $\sigma_1\sigma_2\ldots\sigma_n$ [illegible]

$$(\sigma_1\sigma_2\ldots\sigma_n) \leq (\sigma_1\sigma_2)(\sigma_3\ldots) \cdots (\sigma_{n-1}\sigma_n) \qquad (3)$$

The [illegible] [illegible] [illegible] [illegible] $\sigma_1\sigma_2 \ldots$ [illegible] [illegible] σ_n [illegible] $\phi^{(2)}$ [illegible] [illegible] [illegible] [illegible] [illegible] $\sigma_1\sigma_2\ldots$ [illegible]

$$(6) \qquad s(n) \geq b(n) = \sum (a_1+a_2-1)(a_2+a_3-1) \ldots (a_{i-1}+a_i-1) \quad ,$$

where the sum extends over all compositions of n into an unrestricted number of positive parts.

Eden was able to show that for sufficiently large n, $b(n) > (3.14)^n$ and in this way proved the lower bound given in (3). Klarner [12] later improved this estimate to $(3.21)^n > b(n) > (3.20)^n$, for sufficiently large n. We will now use the method outlined in Chapter 1 to obtain the generating function of $\{b(n)\}$, and at the same time prepare the machinery for treating similar problems.

The sum in (6) has the form of (1.1) with $g(n) = 1$ and $f(m,n) = m+n-1$; in fact, we can suppose the numbers $f(m,n)$ have the form $w(m) + t(n)$; thus, if W and T generate $\{w(n)\}$ and $\{t(n)\}$ respectively, we have

$$(7) \qquad F(x,y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \{w(m) + t(n)\} x^m y^n$$

$$= \frac{yW(x)}{1-y} + \frac{xT(y)}{1-x} \quad .$$

In general, if the kernel of the Fredholm equation in (1.31) has the special form

$$(8) \qquad K(x,y) = \sum_{k=1}^{r} \phi_k(x) \psi_k(y) \quad ,$$

it is said to be of <u>finite rank</u> and the study of the integral equation reduces to a system of linear algebraic equations. (For more details on this point see Riesz and Sz-Nagy [18] , page 161.) Substituting the expression for $F(xy,\frac{1}{s})$ given by (7) into (1.17) we obtain an integral equation having a kernel with finite rank:

$$B(x,y) = G(xy) + W(xy)\, B(x,1) + \frac{1}{2\pi i}\int_c \frac{xyT(\frac{1}{s})\, B(x,s)\, ds}{(1-xy)s} \tag{9}$$

Multiplying equation (9) by $T(1/y)/y$ and integrating we obtain a relation which implies

$$\frac{1}{2\pi i}\int_c T(\tfrac{1}{y})\, B(x,y)\,\frac{dy}{y} = \frac{P(x) + B(x,1)\, Q(x)}{1 - T(x)} , \tag{10}$$

where

$$P(x) = \frac{1}{2\pi i}\int_c T(\tfrac{1}{y})\, G(xy)\,\frac{dy}{y} = \sum_{n=1}^{\infty} t(n)\, g(n)\, x^n , \tag{11}$$

$$Q(x) = \frac{1}{2\pi i}\int_c T(\tfrac{1}{y})\, W(xy)\,\frac{dy}{y} = \sum_{n=1}^{\infty} t(n)\, w(n)\, x^n . \tag{12}$$

The integral representation of sums such as appear in (11) and (12) was probably first discovered by Hadamard (see for example, Titchmarsh [22], pages 157-159). Substituting the expression for the integral given by (10) into (9) and setting $y = 1$, we obtain a linear equation in $B(x,1)$; solving this we obtain

$$B(x,1) = \frac{(1-x)\ G(x)\ (1-T(x)) + xP(x)}{(1-x)(1-T(x))(1-W(x)) - xQ(x)} \tag{13}$$

The relations in (9), (10), and (13) can be combined to find $B(x,y)$ in closed form in terms of W, T, G, P, and Q. When $G(x) = x/(1-x)$, then $P(x) = T(x)$, and (13) reduces to

$$B(x,1) = x/\{(1-x)(1-T(x))(1-W(x)) - xQ(x)\} \quad . \tag{14}$$

An elementary proof of (14) can be given as follows: Substituting $f(m,n) = w(m) + t(n)$ into (1.5), we have for $a < n$,

$$\begin{aligned} b(a,n) &= w(a) \sum_{\nu=1}^{n-a} b(\nu,n-a) + \sum_{\nu=1}^{n-a} t(\nu)\ b(\nu,n-a) \\ &= w(a)\ b(n-a) + \sum_{\nu=1}^{n-a} t(\nu)\ b(\nu,n-a) \quad . \end{aligned} \tag{15}$$

Writing $a-k$ and $n-k$ in place of a and n in (15) we obtain a similar expression for $b(a-k,n-k)$; taking the difference $b(a,n) - b(a-k,n-k)$ and transposing a term gives

$$b(a,n) = b(a-k,n-k) + [w(a) - w(a-k)]\ b(n-a) \ . \tag{16}$$

When $k = a-1$ in (16), we find that each of the numbers $b(a,n)$ can be written in terms of $b(1,\nu)$ and $b(\nu)$, $\nu = 1, 2, \ldots$; thus, for $a < n$,

$$(17) \qquad b(a,n) = b(1,n-a+1) + [w(a) - w(1)]\, b(n-a) \quad .$$

Using the fact that $b(n,n) = g(n) = 1$, we substitute expressions for $b(a,n)$ given by (17) into $b(n) = b(1,n) + b(2,n) + \dots + b(n,n)$ to obtain

$$(18) \qquad b(n) = 1 + \sum_{a=1}^{n-1} b(1,n-a+1) + [w(a) - w(1)]\, b(n-a) \quad .$$

This can be used to show

$$(19) \qquad b(n) - b(n-1) = b(1,n) + \sum_{a=1}^{n-2} [w(n-a) - w(n-a-1)]\, b(a) \quad ,$$

for $n > 2$; when $n = 1$ and 2 we have

$$(20) \qquad b(1) = b(1,1) \ , \qquad \text{and} \qquad b(2) - b(1) = b(1,2) \quad .$$

Equations (19) and (20) imply the following relationship between the generating series:

$$(21) \qquad \sum_{n=1}^{\infty} b(n)x^n - \sum_{n=1}^{\infty} b(n)x^{n+1} = \sum_{n=1}^{\infty} b(1,n)x^n$$

$$+ \sum_{n=3}^{\infty} \sum_{a=1}^{n-2} w(n-a)\, b(a)x^n - \sum_{n=3}^{\infty} \sum_{a=1}^{n-2} w(n-a-1)\, b(a)x^n \quad .$$

Each of the series in (21) can be replaced with the function it represents. Writing ∂B for the partial derivative of $B(x,s)$ with

respect to s at $s = 0$, the result after collecting terms is

(22) $$\partial B = \{w(1) + (1-x)(1-W(x))\}\, B(x,1) \quad .$$

Now we eliminate ∂B from (12). First, setting $a = 1$ in (15) gives

(23) $$b(1,n) = w(1)\, b(n-1) + \sum_{\nu=1}^{n-1} t(\nu)\, b(\nu,n-1) \ ,$$

and substituting expressions for $b(\nu,n-1)$ given by (17) into the sum in the right member of (23) gives

(24) $$b(1,n) = w(1)\, b(n-1) + \sum_{\nu=1}^{n-1} t(\nu)\, b(1,n-\nu)$$

$$+ \sum_{\nu=1}^{n-2} w(\nu)\, t(\nu)\, b(n-\nu-1) - w(1) \sum_{\nu=1}^{n-2} t(\nu)\, b(n-\nu-1) \ ;$$

for $n = 1$ and 2, the relations corresponding to (24) are

(25) $$b(1,1) = 1 \quad \text{and} \quad b(1,2) = w(1)\, b(1) + t(1) \ .$$

Equations (24) and (25) imply the following relationship between the generating functions:

(26) $$\partial B = x + w(1)x\, B(x,1) + T(x)\, \partial B$$

$$- w(1)x\, T(x)\, B(x,1) + xQ(x)\, B(x,1) \quad ,$$

where $Q(x)$ is the function defined in (12). Taken together, the relations in (22) and (26) imply (14).

Now to find the generating function of $\{b(n)\}$ as defined in (6), we put

$$W(x) = T(x) = \sum_{n=1}^{\infty} (n-\tfrac{1}{2})x^n = \frac{x(1+x)}{2(1-x)^2} \quad , \tag{27}$$

$$Q(x) = \sum_{n=1}^{\infty} (n-\tfrac{1}{2})^2 x^n = \frac{x(1+6x+x^2)}{4(1-x)^3} \quad . \tag{28}$$

Substituting these functions into (14) gives

$$B(x,1) = \frac{x(1-x)^3}{1-5x+7x^2-4x^3} = \sum_{n=1}^{\infty} b(n)x^n \quad . \tag{29}$$

Multiplying equation (29) by $1-5x+7x^2-4x^3$ and equating coefficients in the resulting identity gives $b(1) = 1$, $b(2) = 2$, $b(3) = 6$, $b(4) = 19$, and

$$b(n+3) = 5b(n+2) - 7b(n+1) + 4b(n) \quad , \tag{30}$$

for $n = 2, 3, \ldots$. Since the largest real root of the auxiliary equation for the difference equation in (30) lies between 3.20 and 3.21 , we conclude that $(3.20)^n < b(n) < (3.21)^n$ for sufficiently large n .

The method just used to obtain a lower bound for n-ominoes also applies to animals with cells of different shapes. For example, animals with connected strips of hexagons in each row are enumerated by

$$b(n) = \sum (a_1+a_2)(a_2+a_3) \ldots (a_{i-1}+a_i) \quad , \tag{31}$$

where the sum extends over all compositions of n. This follows since a strip of r hexagons can be connected along the upper edge of a strip of s hexagons in $r+s$ ways.

To find the generating function for the sequence $\{b(n)\}$ defined by (31) we substitute

$$W(x) = T(x) = \sum_{n=1}^{\infty} nx^n = \frac{x}{(1-x)^2} \quad , \quad \text{and} \tag{32}$$

$$Q(x) = \sum_{n=1}^{\infty} n^2 x^n = \frac{x(1+x)}{(1-x)^3} \quad , \tag{33}$$

into equation (14) to obtain

$$B(x,1) = \frac{x(1-x)^3}{1-6x+10x^2-7x^3+x^4} = \sum_{n=1}^{\infty} b(n)x^n \quad . \tag{34}$$

The relation in (34) implies $b(1) = 1$, $b(2) = 3$, $b(3) = 11$, $b(4) = 42$, and

$$b(n+4) = 6b(n+3) - 10b(n+2) + 7b(n+1) - b(n) \quad , \tag{35}$$

for $n = 1, 2, \ldots$. Furthermore, the largest real root of the auxiliary equation for the difference equation in (35) lies between 3.87 and 3.88, so that for sufficiently large n, $(3.88)^n > b(n) > (3.87)^n$.

Golomb [3] suggested the problem of determining the number of incongruent n celled animals with hexagonal cells; these numbers are 1, 1, 3, 7, 22, and 83 for $n = 1, 2, 3, 4, 5,$ and 6 respectively. They correspond to free animals while the numbers defined in (31) correspond to fixed animals of a certain type. Since a hexagon has 12 symmetries in the plane, the numbers of fixed and free animals with hexagonal cells differ by a factor of 12 at most.

A rhombus is formed when two equilateral triangles are joined along an edge; thus, a lower bound for the number of animals with n rhomboidal cells can be used to find a lower bound for the number of animals with $2n$ triangular cells. A connected strip of r rhombuses can be joined above a strip of s rhombuses in $2(r+s-1)$ ways, since the strip of r rhombuses has two orientations with respect to a reflection about its midsection. Thus, a lower bound for the number of fixed animals with n rhomboidal cells is

$$(36) \qquad b(n) = \sum (2a_1+2a_2-2)(2a_2+2a_3-2) \ldots (2a_{i-1}+2a_i-2) \;,$$

where the sum extends over all compositions of n. Now to find the generating function of $\{b(n)\}$ as defined by (36), we can obtain the appropriate expression for $W(x)$ and $Q(x)$ by multiplying equations

(27) and (28) by 2 and 4 respectively. Substituting these functions into (14) gives

(37) $$B(x,1) = \frac{x(1-x)^3}{1-6x+8x^2-6x^3-x^4} = \sum_{n=1}^{\infty} b(n)x^n .$$

Of course, (37) implies $b(1) = 1$, $b(2) = 3$, $b(3) = 13$, $b(4) = 59$, and

(38) $$b(n+4) = 6b(n+3) - 8b(n+2) + 6b(n+1) + b(n) ,$$

for $n = 1, 2, \ldots$. The auxiliary equation for (38) has its largest real root between 4.54 and 4.55, so for sufficiently large n, $(4.55)^n > b(n) > (4.54)^n$; from this we can conclude that the number of fixed animals with n triangular cells is greater than $(2.13)^n$ for sufficiently large n .

CHAPTER III

IMPROVED LOWER BOUNDS FOR THE CELL GROWTH PROBLEM

In the last chapter we defined $F_{a_1a_2\ldots a_i}$, for each composition $(a_1,a_2,\ldots,a_i)$ of n, as the set of equivalence classes in S_n containing n-ominoes with a_j cells in the j^{th} row of the n-omino, for $j = 1,2,\ldots,i$. We used the fact that a strip of s cells can be joined along the upper edge of a strip of r cells in $r+s-1$ ways; thus, there are exactly $(a_1+a_2-1)(a_2+a_3-1)\ldots$ elements in the subset $F^*_{a_1a_2\ldots a_i}$ of $F_{a_1a_2\ldots a_i}$ containing n-ominoes with a single strip of cells in each row. This gives

$$(1) \qquad f^*(a_1,a_2,\ldots,a_i) = f^*(a_1,a_2)f^*(a_2,a_3)\ldots f^*(a_{i-1},a_i) \quad ,$$

where $f^*p,q,\ldots)$ denotes the number of elements in $F^*_{pq\ldots}$.

As the relation in (1) suggests, it is easy to give a one-one correspondence between the elements of $F^*_{a_1a_2\ldots a_i}$ and the $(i-1)$-tuples of $F^*_{a_1a_2} \times F^*_{a_2a_3} \times \ldots \times F^*_{a_{i-1}a_i}$. To do this, suppose x is in standard position and is an element of the equivalence class X in $F^*_{a_1a_2\ldots a_i}$. The cells of x in the j^{th} and $(j+1)^{st}$ rows above the x-axis

comprise a representative element y_j of an equivalence class Y_j in $F^*_{a_j a_{j+1}}$, for $j = 1,2,\ldots,i-1$. In this way, X corresponds to a unique $(i-1)$-tuple $(Y_1,Y_2,\ldots,Y_{i-1})$. On the other hand, starting with an element $(Y_1,Y_2,\ldots,Y_{i-1})$ of $F^*_{a_1 a_2} \times F^*_{a_2 a_3} \times \ldots \times F^*_{a_{i-1} a_i}$, we construct x as follows: Suppose $y_1 \in Y_1$ is in standard position and then choose $y_j \in Y_j$ so that the a_j cells in the bottom row of y_j are the same as the a_j cells in the top row of y_{j-1}, for $j = 2,3,\ldots,i-1$. Now $\bigcup_{j=1}^{i-1} y_j = x$ is a representative element of X. These obvious constructions establish a one-one correspondence between the elements of the two sets under discussion.

Now we are going to show that in fact there is a one-one correspondence between all the elements of $F_{a_1 a_2} \times F_{a_2 a_3} \times \ldots \times F_{a_{i-1} a_i}$ and a certain subset of $F_{a_1 a_2 \ldots a_i}$.

<u>Theorem 1</u>: $\quad f(a_1,a_2,\ldots,a_i) \geq f(a_1,a_2)f(a_2,a_3)\ldots f(a_{i-1},a_i)$.

<u>Proof:</u> We call an $r \times 1$ rectangle located in a column of the square lattice and a non-empty subset of the r cells contained in this rectangle an <u>r-component</u>. For example, an n-omino in an equivalence class in $F_{a_1 a_2 \ldots a_r}$ is a sequence of r-components contained in an $r \times j$ rectangle.

Now suppose an $r \times j$ rectangle contains a sequence R of r-components such that the top row of the rectangle contains exactly

c cells belonging to the components of R ; also, suppose an $s \times k$ rectangle contains a sequence S of s-components such that the bottom row of the rectangle contains exactly c cells belonging to the components of S . Clearly, we can translate the components to the right in either or both of the rectangles leaving gaps between the components so that the c cells in the bottom row of S can be made to cover the c cells in the top row of R . The sequence $R*S$ of (r+s-1)-components which results when the components of R and S are joined in this way is called the sum of R and S .

Let $(Y_1, Y_2, \ldots, Y_{i-1})$ be a given element of $F_{a_1a_2} \times F_{a_2a_3} \times \ldots \times F_{a_{i-1}a_i}$ and suppose y_j is a representative element of Y_j , for $j = 1,2,\ldots,i-1$. We are going to construct a representative element y of an equivalence class $Y = Y(Y_1, Y_2, \ldots, Y_{i-1})$ of $F_{a_1a_2\ldots a_i}$ such that the sequence of 2-components in the j^{th} and $(j+1)^{st}$ rows of y is the same as the sequence of 2-components of y_j , for $j = 1,2,\ldots,i-1$.

Consider the sequence of i-components given by $((\ldots((y_1*y_2)*y_3)*\ldots)*y_{i-1}) = y_1*y_2*\ldots*y_{i-1}$; if this sequence is not an n-omino, 2-components containing disconnected cells can be translated to the left and joined to cells they were joined to formerly in $y_1, y_2, \ldots, y_{i-1}$, it being understood that overlapping 2-components must be translated simultaneously. The n-omino formed in this way has the desired properties; since, different sequences $(Y_1, Y_2, \ldots, Y_{i-1})$ of

$F_{a_1a_2} \times F_{a_2a_3} \times \ldots \times F_{a_{i-1}a_i}$ give rise to different elements Y of $F_{a_1a_2\ldots a_i}$, the theorem is proved. An example of the construction is given in Figure 1 .

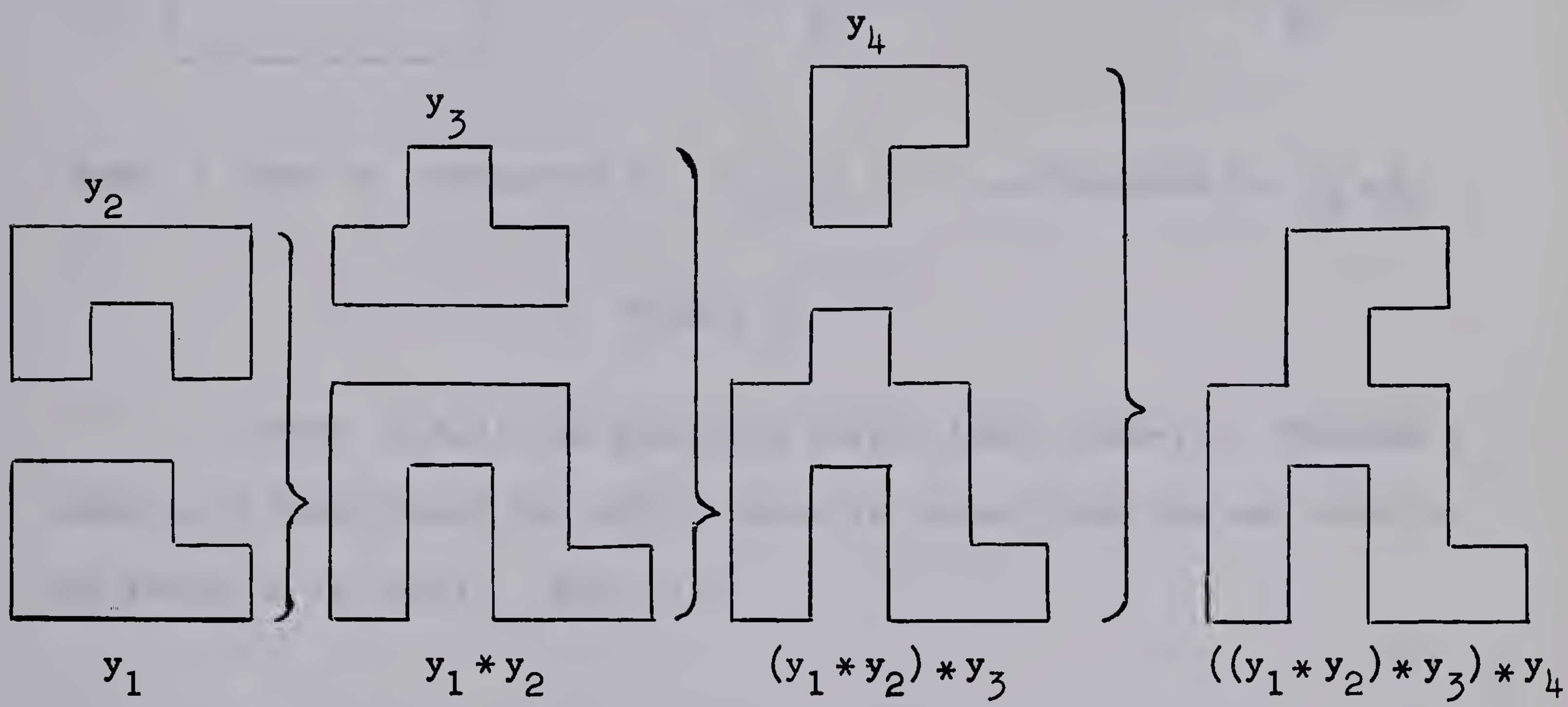
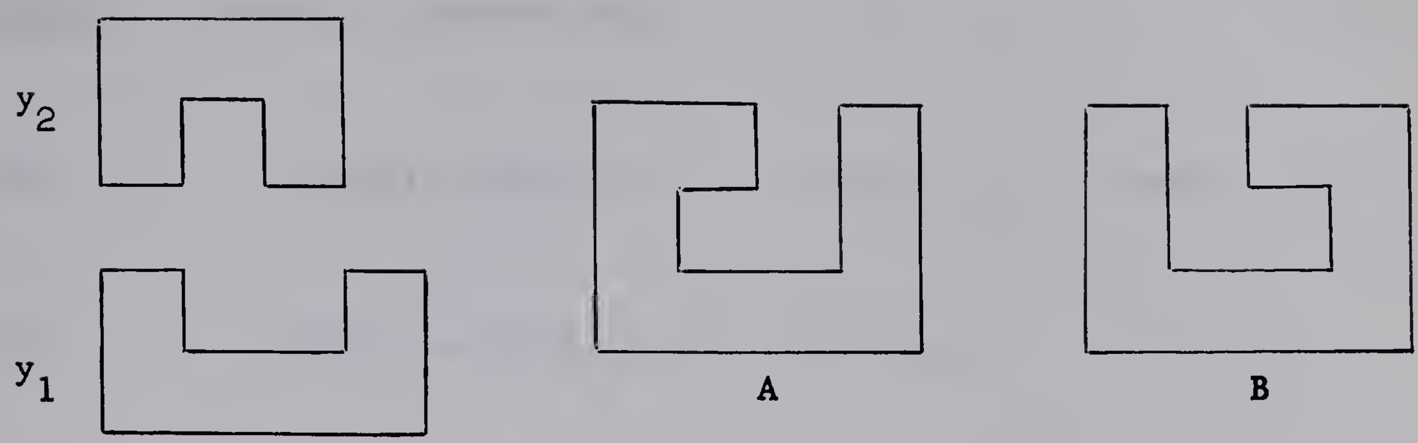


Figure 1.

We note that when $y_1 * y_2 * \ldots * y_{i-1}$ (described in the proof of Theorem 1) is disconnected, there may be many ways in which the cells can be translated to the right or the left to eventually form an n-omino; thus, n-ominoes in different equivalence classes in $F_{a_1a_2\ldots a_i}$ may have the same sequence of 2-components in their j^{th} and $(j+1)^{st}$ rows, for $j = 1,2,\ldots,i-1$. An example of this is given in Figure 2 .



Both A and B correspond to (y_1, y_2) ; A corresponds to $y_1 * y_2$.

Figure 2.

Since $f(m,n)$ is generally larger than $(m+n-1)$, Theorem 1 leads to a lower bound for $s(n)$ which is larger than the one given by the relation in (2.6) . Thus, if

$$(2) \qquad b(n) = \sum f(a_1,a_2)f(a_2,a_3)\ldots f(a_{i-1},a_i) \quad ,$$

where the sum extends over all compositions $(a_1,a_2,\ldots,a_i)$ of n , then $s(n) \geq b(n)$, for $n = 1,2,\ldots$. Once we have the generating function of $\{f(m,n)\}$, the methods given in Chapter I can be applied to the sum in (2) .

<u>Theorem 2</u>:

$$(3) \qquad H(x,y) = \frac{1-xy}{1-x-y+x^2y^2} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} f(m,n)x^m y^n \quad .$$

Proof: First, we observe that

(4) $$f(n,0) = f(0,n) = 1 , \quad n = 0,1,..., \quad \text{and}$$

(5) $$f(n,1) = f(1,n) = n , \quad n = 1,2,... \quad .$$

Next, we show that

(6) $$f(m+2,n+2) = f(m+2,n+1) + f(m+1,n+2) - f(m,n) \quad ,$$

for $m,n = 0,1,...$. There is a way to construct a representative element of each equivalence class in $F_{m+2,n+2}$ by adding a cell to representative elements in each of the equivalence classes in $F_{m+2,n+1} \cup F_{m+1,n+2}$ so that exactly $f(m,n)$ elements are duplicated in the process. To do this suppose x is in standard position and is an element of an equivalence class in $F_{m+2,n+1}$ (or in $F_{m+1,n+2}$), and join a new cell to x in the second row (or first row) so that the new cell is a maximum distance to the right of the origin. It is easy to see that the $(n+m+4)$-ominoes obtained in this way are distinct except those which can be obtained by connecting a 2×2 block to the right end of elements in standard position selected from the equivalence classes of F_{mn} ; these $(m+n+4)$-ominoes will be constructed once in connection with an element of $F_{m+1,n+2}$ and again in connection with an element of $F_{m+2,n+1}$.

In order to verify that $H(x,y)$ generates $\{f(m,n):m,n = 0,1,... \}$

we check that $H(x,y)$ and $\partial H(x,y)/\partial y$ respectively generate $\{f(m,0)\}$ and $\{f(m,1)\}$ at $y = 0$; similarly, we see that $H(x,y)$ and $\partial H(x,y)/\partial x$ generate $\{f(0,n)\}$ and $\{f(1,n)\}$ at $x = 0$. Multiplying relation (3) by $(1-x-y+x^2y^2)$ and equating coefficients of $x^m y^n$, we see that (3) implies (6). This completes the proof of Theorem 2.

Using (4), (5), and (6) we can compute a table of values of $f(m,n)$; since $f(m,n) = f(n,m)$, it is only necessary to list $f(m,n)$ with $m \geq n$. The University of Alberta Computing Department found $f(m,n)$ for $m+n \leq 30$, and computed $b(a,n)$, $a = 1,2,\ldots,n$, and $b(n)$, $n = 1,2,\ldots,30$, by means of (1.5); these numbers are as follows:

n	b(n)	n	b(n)
1	1	16	82830896
2	2	17	305299746
3	6	18	1126742091
4	19	19	4162932345
5	63	20	15395001510
6	216	21	56977972065
7	756	22	211024241503
8	2681	23	782014829839
9	9600	24	2899482522068
10	34626	25	10755229416160
11	125582	26	39910544174211
12	457425	27	148150485556469
13	1671854	28	550107559124955
14	6127385	29	2043174182977793
15	22507654	30	7590365195768629

Theorem 2 gives

$$(7) \qquad F(x,y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} f(m,n)x^m y^n = H(x,y) - \frac{(1-xy)}{(1-x)(1-y)} \;;$$

furthermore, we know from the discussion in Chapter 1 that $B(x,y)$, the generating function of $\{b(a,n)\}$ defined by (2), satisfies the integral equation

$$(8) \qquad B(x,y) = \frac{xy}{1-xy} + \frac{1}{2\pi i} \int_c F(xy, \tfrac{1}{s})\, B(x,s)\, \frac{ds}{s} \;.$$

Although the kernal of the integral in (8) is a rational function, the solution function seems to be a much more complicated function, but we have not been able to find $B(x,y)$ in closed form so this can only be surmised by finding the functions $B_1, B_2, \ldots$ described in (1.13), (1.14), and (1.15). The following general technique can be used to estimate $b(n)$ from below. First, we select a function $F^*(x,y)$ which generates $\{f^*(m,n)\}$ such that $0 \leq f^*(m,n) \leq f(m,n)$, $m,n = 1,2,\ldots,$ and define a number $b^*(n)$ by substituting f^* for f in (2); of course, $b^*(n) \leq b(n)$. Furthermore, we choose $F^*(x,y)$ so that it is possible to obtain the solution function (in closed form) of the integral equation

$$(9) \qquad B^*(x,y) = \frac{xy}{1-xy} + \frac{1}{2\pi i} \int_c F^*(xy, \tfrac{1}{s})\, B^*(x,s)\, \frac{ds}{s} \;,$$

where $B^*(x,y)$ generates $\{b^*(a,n)\}$.

In Chapter I we used

$$(10) \qquad f^*(m,n) = \begin{cases} f(m,n) , & \text{if} \quad m,n \le j \\ 0 , & \text{otherwise} \end{cases}$$

to define $B^j(x,y) = B^*(x,y)$, and noted that $B^j(x,1)$ is a rational function with its demominator equal to $\Delta_j(x)$ defined in (1.25) and (1.26) . The polynomials $(-1)^{j+1}x^{j(j+1)/2}\Delta_j(\frac{1}{x})$, for $j = 1,2,\ldots,6$, are as follows:

(11) $\quad x-1$

(12) $\quad x^3-x^2-3x-1$

(13) $\quad x^6-x^5-3x^4-10x^3+2x+1$

(14) $\quad x^{10}-x^9-3x^8-10x^7-27x^6+13x^5+18x^4+29x^3+9x^2-3x-2$

(15) $\quad x^{15}-x^{14}-3x^{13}-10x^{12}-27x^{11}-70x^{10}+76x^9+134x^8+242x^7+203x^6$
$\qquad +14x^5-20x^4+120x^3-9x^2-x+1$

(16) $\quad x^{21}-x^{20}-3x^{19}-10x^{18}-27x^{17}-70x^{16}-183x^{15}+357x^{14}+730x^{13}$
$\qquad +1413x^{12}+1743x^{11}+972x^{10}-222x^9-665x^8-1001x^7$
$\qquad -409x^6-96x^5+378x^4+129x^3-5x^2+x-3$

From the discussion on page 9 of Chapter I, we can conclude that if θ_j is the largest real root of $x^{j(j+1)/2}\,\Delta_j(\frac{1}{x}) = 0$, then $b(n) > (\theta_j-\epsilon)^n$, for every $\theta_j > \epsilon > 0$, and for all sufficiently large n. Using (11)-(16) we find that $\theta_1 = 1$, $\theta_2 > 2.41$, $\theta_3 > 3.00$, $\theta_4 > 3.34$, $\theta_5 > 3.50$, and $\theta_6 > 3.59$; thus,

$$s(n) > b(n) > (3.59)^n \quad , \tag{17}$$

for all sufficiently large n.

There is still another means of approximating $B(x,y)$ which has some advantages over the method just used, but this method depends strongly on the form of $F(x,y)$. We can write

$$F(x,y) = \sum_{k=1}^{\infty} F_k(x)y^k = \sum_{k=1}^{\infty} \frac{f_k(x)y^k}{(1-x)^{k+1}} \quad , \tag{18}$$

where $f_k(x)$ is a polynomial of degree $M = M(k)$. This implies that for fixed k and all $m > M$,

$$\sum_{r=0}^{k+1} (-1)^r \binom{k+1}{r} f(m-r,k) = 0 \quad ; \tag{19}$$

furthermore, for $m > \max\,\{M(1),\ldots,M(k)\}$ each of the sequences $\{f(m,1)\}, \ldots, \{f(m,k)\}$ satisfies the difference equation (19).

For a fixed j we define

$$(20) \qquad F^*(x,y) = \sum_{k=1}^{\infty} F_k^*(x)y^k = \sum_{m,n=1}^{\infty} f^*(m,n)x^m y^n \quad ,$$

where

$$(21) \qquad F_k^*(x) = \begin{cases} F_k(x) \ , & \text{for} \quad k \leq M(j) \\ \sum_{i=1}^{j} (-1)^{i+1} \binom{j}{i} F_{j-i}^*(x), & \text{for} \quad k > M(j) \end{cases} \quad .$$

Combining (20) and (21) we have

$$(22) \qquad F^*(x,y) = \frac{\sum_{\nu=1}^{M} \left\{ \sum_{i=0}^{\nu-1} (-1)^i \binom{j}{i} F_{\nu-i}(x) \right\} y^{\nu}}{(1-y)^j}$$

It follows from these definitions that

$$(23) \qquad F(x,y) - F^*(x,y) = \sum_{m=M+1}^{\infty} \sum_{n=j+1}^{\infty} \{f(m,n)-f^*(m,n)\} \, x^m y^n \quad ,$$

so that evidently F^* tends to F (and hence B^* tends to B) as j tends to infinity. It also turns out that $0 < f^*(m,n) \leq f(m,n)$, for $m,n = 1,2,\ldots,$ and $j = 1,2,3$. This can be checked by writing $F-F^*$ as a rational function to find a difference equation satisfied by the numbers $\{f(m,n)-f^*(m,n)\}$; the fact that these numbers are non-negative can then be established by an induction argument. It may be that $0 < f^*(m,n) \leq f(m,n)$, for $j > 3$, however, we do not need this condition.

If we put $j = 3$ in (22) and use the resulting function $F^*(x,y)$ as the kernal in

$$B^*(x,y) = \frac{xy}{1-xy} + \frac{1}{2\pi i} \int_c F^*(xy, \tfrac{1}{s})\, B^*(x,s)\, \frac{ds}{s} , \tag{24}$$

the integral equation becomes

$$\begin{aligned} B^*(x,y) = \frac{xy}{1-xy} &+ \{3F_1(xy)-3F_2(xy)+F_3(xy)\}B^*(x,1) \\ &+ \{-2F_1(xy)+3F_2(xy)-F_3(xy)\}\, \partial B^*(x,1) \\ &+ \{F_1(xy)-2F_2(xy)+F_3(xy)\}\, \frac{\partial^2 B(x,1)}{2} , \end{aligned} \tag{25}$$

where $\partial^i B(x,1)$ denotes $\partial^i B(x,s)/\partial s^i$ evaluated at $s = 1$. Letting $Q_1(x,y)$, $Q_2(x,y)$, and $Q_3(x,y)$ denote the coefficients of $B^*(x,1)$, $\partial B^*(x,1)$, and $\partial^2 B^*(x,1)/2$ respectively, we take partial derivatives with respect to y at $y = 1$ in (25) to obtain a system of equations equivalent to the following:

$$\frac{-x}{1-x} = \{Q_1(x,1)-1\}B^*(x,1)+Q_2(x,1)\partial B^*(x,1)+Q_3(x,1)\, \frac{\partial^2 B^*(x,1)}{2} , \tag{26}$$

$$\frac{-x}{(1-x)^2} = \partial Q_1(x,1)B^*(x,1)+\{\partial Q_2(x,1)-1\}\partial B^*(x,1)+\partial Q_3(x,1)\, \frac{\partial^2 B^*(x,1)}{2} \tag{27}$$

$$\frac{-x^2}{(1-x)^3} = \frac{\partial^2 Q_1(x,1)}{2}\, B^*(x,1)+\frac{\partial^2 Q_2(x,1)}{2}\, \partial B^*(x,1)+\left\{ \frac{\partial^2 Q_3(x,1)}{2} -1\right\} \frac{\partial^2 B^*(x,1)}{2} . \tag{28}$$

Applying Cramer's rule to this system to solve for $B^*(x,1)$, we obtain a rational function having its denominator equal to $\Delta(x)(1-x)^{12}$ where

$$(29) \qquad \Delta(x) = \begin{vmatrix} F_1 - 1 & F_2 - 1 & F_3 - 1 \\ \partial F_1 - 1 & \partial F_2 - 2 & \partial F_3 - 3 \\ \frac{\partial^2 F_1}{2} & \frac{\partial^2 F_2}{2} - 1 & \frac{\partial^2 F_3}{2} - 3 \end{vmatrix} ,$$

and $\partial^i F_j = \left. \frac{\partial^i F_j(xy)}{\partial y^i} \right|_{y=1}$. Substituting the values of $F_1, \partial F_1, \ldots$ given by (7), we expand (29) to find that $\Delta(x)(1-x)^{12}$ is equal to

$$(30) \qquad 3x^{13}-15x^{12}+63x^{11}-143x^{10}+288x^9-662x^8+1101x^7-1261x^6+1032x^5 -612x^4+260x^3-75x^2+13x-1 .$$

The largest real root of $\Delta(1/x)(1-1/x)^{12} = 0$ is between 3.60 and 3.61; hence,

$$(31) \qquad s(n) \geq b(n) > b^*(n) > (3.60)^n ,$$

for all sufficiently large n .

If regular hexagons of unit area are used as cells of an animal, the number of "two row animals" (analogous to the elements of F_{mn}) of this type with m cells in the first row and n cells in the

second is exactly $\binom{m+n}{n}$. By modifying the proof of Theorem I slightly, we can show that there are more than

$$(32) \qquad b(n) = \sum \binom{a_1+a_2}{a_2}\binom{a_2+a_3}{a_3}\cdots\binom{a_{i-1}+a_i}{a_i}$$

"fixed" animals with n hexagonal cells, where the sum in (32) extends over all compositions $(a_1,a_2,\ldots,a_i)$ of n . It follows from (5.3) and (5.10) that $b(n) \geq p(n+2) = \frac{1}{n+1}\binom{2n}{n}$. Thus, for every $0 < \epsilon < 4$,

$$(33) \qquad b(n) > (4-\epsilon)^n \quad ,$$

for all sufficiently large n .

CHAPTER IV

READ'S THEOREM

We mentioned in Chapter II that Read [17] enumerated the number of equivalence classes in S_n containing n-ominoes with cells in exactly k rows of the plane. He characterized such an n-omino as a sequence of k-components which satisifies certain restrictions: Some k-components are permitted to begin a sequence, each k-component i may be succeeded in the sequence only by k-components selected from a set X_i, and only certain k-components may terminate a sequence. In what follows, we reformulate the problem which Read posed and solved, and give an alternative solution based on the methods outlined in Chapter I.

Let $X = \{1, 2, \ldots, k\}$ be a given finite set of objects and suppose Y and Z are non-empty subsets of X. Furthermore, suppose that each object i in X has assigned to it a subset X_i of X and an integer $w(i)$. We let $t(q,n)$ denote the number of q-tuples $(a_1,a_2,\ldots,a_q)$, $a_i \in X$, $i = 1, 2, \ldots, q$, such that (i) $w(a_1) + \ldots + w(a_q) = n$, (ii) $a_1 \in Y$, (iii) $a_q \in Z$, and (iv) $a_i \in X_{a_{i-1}}$, $i = 2, 3, \ldots, q$. Now

(1) $$T_q(z) = \sum_n t(q,n)z^n \quad , \qquad \text{and}$$

(2) $$T(s,z) = \sum_q \sum_n t(q,n)s^q z^n$$

are called the $\underline{q^{th}}$ <u>counting polynomial</u> and the <u>sequence counting series</u> respectively. Read proved the following theorem which gives a means of calculating $T_q(z)$ and $T(s,z)$.

<u>Theorem</u> (R. C. Read):

(3) $$T_q(z) = P[\psi(z)]^{q-1}S \quad , \qquad \text{and}$$

(4) $$T(s,z) = sP[I-s\psi(z)]^{-1}S \quad ,$$

where I is the $k \times k$ identity matrix, S is the column vector $\{s_1,s_2,\ldots,s_k\}$, P is the row vector $\{p_1,p_2,\ldots,p_k\}$, and $\psi(z)$ is a $k \times k$ matrix $[f(m,n)]$ defined by

(5) $$s_n = \begin{cases} z^{w(n)} , & \text{if } n \in Y \\ 0 , & \text{otherwise} \end{cases} \quad ,$$

(6) $$p_n = \begin{cases} 1 , & \text{if } n \in Z \\ 0 , & \text{otherwise} \end{cases} \quad , \qquad \text{and}$$

$$f(m,n) = \begin{cases} z^{w(n)}, & \text{if } n \in X_m \\ 0, & \text{otherwise} \end{cases} \quad . \tag{7}$$

We will not present Read's proof of his theorem, but proceed at once in deriving a second solution of his problem. First, using the definitions of $f(m,n)$ and p_n given in (6) and (7) we define

$$F(x,y:z) = \sum_{m=1}^{k} \sum_{n=1}^{k} f(m,n)\, x^m y^n , \quad \text{and} \tag{8}$$

$$G(x) = \sum_{n=1}^{k} p_n x^n , \tag{9}$$

and then put $sF(x,y:z)$ and $sG(x)$ in place of $F(x,y)$ and $G(x)$ in (1.7) to define $b(n)$ as in (1.1) . Now consider the generating function of the sequence $\{b(a,n)\}$ defined in this way:

$$B(x,y:s,z) = \sum_{n=1}^{\infty} \sum_{a=1}^{k} b(a,n) y^a x^n \quad . \tag{10}$$

If $(a_1,a_2,\ldots,a_q)$ is a composition of n which satisfies the four properties used to define $t(q,n)$, then it contributes $s^q z^{w(a_2)+w(a_3)+\ldots+w(a_q)}$ to the coefficient of $y^{a_1} x^n$ in (10); otherwise, the contribution is 0 . We want the sum of the generating functions of $\{z^{w(a)} b(a,n) : n = 1,2,\ldots\}$ for $a \in Y$; thus,

$$T(s,z) = \sum_{n=1}^{k} \frac{s_n}{n!} \frac{\partial^n}{\partial y^n} \{B(1,y:s,z)\} \Big|_{y=0} \quad . \tag{11}$$

We use the relations in (1.24), (1.25), and (1.26) to find the partial derivatives in (11). Let $\Phi(s,z) = [sf(m,n)]$, let I denote the identity matrix of order k, let $\Delta = \Phi(s,z) - I$, and let Δ_n denote the matrix obtained by replacing the n^{th} column of Δ with the column vector $\{-sp_1, -sp_2, \ldots, -sp_k\}$. This gives

$$T(s,z) = \sum_{n=1}^{k} s_n \det \Delta_n / \det \Delta \quad . \tag{12}$$

The following example illustrates the use of Read's Theorem and relation (12). Letting 1, 2, and 3 denote the 2-components with one cell in the top row, with one cell in the bottom row, and with two cells respectively, we put $w(1) = w(2) = 1$, $w(3) = 2$, $X_1 = \{1,3\}$, $X_2 = \{2,3\}$, $X_3 = Y = Z = \{1,2,3\}$; now $t(q,n)$ is the number of n-ominoes which may be constructed as a sequence of q 2-components. Read used his theorem to show that in this case

$$T(s,z) = [s,s,s] \begin{bmatrix} sz-1 & 0 & sz^2 \\ 0 & sz-1 & sz^2 \\ sz & sz & sz^2-1 \end{bmatrix}^{-1} \begin{bmatrix} z \\ z \\ z^2 \end{bmatrix} \tag{13}$$

$$= \frac{2sz + sz^2 + s^2z^3}{1 - sz - sz^2 - s^2z^3} \quad .$$

Using relation (12) we find

$$(14) \qquad \det \Delta = \begin{vmatrix} sz-1 & 0 & sz^2 \\ 0 & sz-1 & sz^2 \\ sz & sz & sz^2-1 \end{vmatrix} = (sz-1)(1-sz-sz^2-s^2z^3) \quad ,$$

$$(15) \qquad \det \Delta_1 = \begin{vmatrix} -s & 0 & sz^2 \\ -s & sz-1 & sz^2 \\ -s & sz & sz^2-1 \end{vmatrix} = s(sz-1) \quad ,$$

$$(16) \qquad \det \Delta_2 = \begin{vmatrix} sz-1 & -s & sz^2 \\ 0 & -s & sz^2 \\ sz & -s & sz^2-1 \end{vmatrix} = s(sz-1) \quad ,$$

$$(17) \qquad \det \Delta_3 = \begin{vmatrix} sz-1 & 0 & -s \\ 0 & sz-1 & -s \\ sz & sz & -s \end{vmatrix} = s(sz-1)(sz+1) \quad ,$$

so that combining these we obtain

$$(18) \qquad T(s,z) = \{z \det \Delta_1 + z \det \Delta_2 + z^2 \det \Delta_3\}/\det \Delta$$

$$= \frac{2sz + sz^2 + s^2z^3}{1 - sz - sz^2 - s^2z^3} \quad .$$

The method outlined in Chapter I can also be used to find sequences $\{t^*(q,n): q,n = 1,2,\ldots\}$ and $\{t_*(q,n): q,n = 1,2,\ldots\}$,

such that

$$t_*(q,n) \leq t(q,n) \leq t^*(q,n) \quad , \tag{19}$$

for $q,n = 1,2,\ldots$. To do this we let W_i denote the subset of X containing all objects with weight i , and define

$$f^*(m,n) = \max_{j \in W_m} |X_j \cap W_n| \quad , \qquad \text{and} \tag{20}$$

$$f_*(m,n) = \min_{j \in W_m} |X_j \cap W_n| \quad , \tag{21}$$

where $|S|$ denotes the number of elements in set S . Also, we define

$$g^*(n) = \begin{cases} 1, & \text{if } W_n \cap Z \neq \phi \\ 0, & \text{otherwise} \end{cases} \quad , \qquad \text{and} \tag{22}$$

$$g_*(n) = \begin{cases} 1, & \text{if } W_n \subseteq Z \\ 0, & \text{otherwise} \end{cases} \quad , \tag{23}$$

and let

$$b^*(a,n) = \sum s^i f^*(a_1,a_2)\, f^*(a_2,a_3) \ldots f^*(a_{i-1},a_i)\, g^*(a_i) \quad , \tag{24}$$

$$b_*(a,n) = \sum s^i f_*(a_1,a_2)\, f_*(a_2,a_3) \ldots f_*(a_{i-1},a_i)\, g_*(a_i) \; , \tag{25}$$

where the sums in (24) and (25) extend over all compositions $(a_1,a_2,\ldots,a_i)$

of n with $a_1 = a$. The generating functions

(26) $$B^*(x,y:s) = \sum_{a,n} b^*(a,n)y^a x^n \quad , \quad \text{and}$$

(27) $$B_*(x,y:s) = \sum_{a,n} b_*(a,n)y^a x^n \quad ,$$

are defined by

(28) $$sF^*(x,y) = \sum_{m,n} f^*(m,n)sx^m y^n \ , \quad sG^*(x) = \sum_{n} g^*(n)sx^n \ , \quad \text{and}$$

(29) $$sF_*(x,y) = \sum_{m,n} f_*(m,n)sx^m y^n \ , \quad sG_*(x) = \sum_{n} g_*(n)sx^n \quad ,$$

respectively, just as $B(x,y)$ is defined by $F(x,y)$ and $G(x)$ in (1.1). Letting $s(a)$ denote the number of objects in Y with weight a , we define

(30) $$T^*(s,z) = \sum_{a} \frac{s(a)}{a!} \frac{\partial^a}{\partial y^a} \{B^*(z,y:s)\}_{y=0} \quad , \quad \text{and}$$

(31) $$T_*(s,z) = \sum_{a} \frac{s(a)}{a!} \frac{\partial^a}{\partial y^a} \{B_*(z,y:s)\}_{y=0} \quad .$$

The sequences $\{t^*(q,n)\}$ and $\{t_*(q,n)\}$ generated by (30)

and (31) respectively, satisfy (19); this follows from (24) and (25), and the definitions of f^*, g^*, f_*, and g_*. Suppose $(a_1, a_2, \ldots, a_q)$ is a composition of n and consider its contribution to the sums in (24) and (25): There are at most $s(a_1)f^*(a_1,a_2)\ldots f^*(a_{q-1},a_q)g^*(a_q)$ and at least $s(a_1)f_*(a_1,a_2)f_*(a_2,a_3)\ldots f_*(a_{q-1},a_q)g_*(a_q)$ sequences of objects $(0_1, 0_2, \ldots, 0_q)$ such that $w(0_i) = a_i$, for $i = 1,2,\ldots,q$. The coefficient of $s^q z^n$ in (30) and (31) is

$$(32) \qquad t^*(q,n) = \sum s(a_1)\ f^*(a_1,a_2)\ f^*(a_2,a_3)\ \ldots\ f^*(a_{q-1},a_q)\ g^*(a_q) ,$$

and

$$(33) \qquad t_*(q,n) = \sum s(a_1)\ f_*(a_1,a_2)\ f_*(a_2,a_3)\ \ldots\ f_*(a_{q-1},a_q)\ g_*(a_q) ,$$

respectively, where the sums in (32) and (33) extend over all compositions of n into exactly q parts.

The sums in (30) and (31) are finite, and the partial derivatives involved can be found in terms of determinants as shown in (1.24) - (1.26). From this we can conclude that $T^*(s,z)$ and $T_*(s,z)$ are rational functions just as $T(s,z)$ is. Furthermore, if $f^*(m,n) = f_*(m,n)$ and $g^*(n) = g_*(n)$, for $m,n = 1,2,\ldots,$ then $T^*(s,z) = T_*(s,z) = T(s,z)$. This last fact can sometimes be used to greatly reduce the work involved in calculating $T(s,z)$ by means of Read's Theorem or the formula given in (12).

The example treated in (13) - (18) is a case for which $T^*(s,z) = T_*(s,z) = T(s,z)$: This follows since all of the objects are weighted 1 or 2, and $f^*(1,1) = f_*(1,1) = 1$, $f^*(1,2) = f_*(1,2) = 1$, $f^*(2,1) = f_*(2,1) = 2$, $f^*(2,2) = f_*(2,2) = 1$, and any of the objects may begin or end a sequence. Calculating $T(s,z)$ by means of (30) and (1.24) - (1.26) we obtain:

$$(34) \qquad T(s,z) = \frac{2\begin{vmatrix} -sz & sz \\ -sz^2 & sz^2-1 \end{vmatrix} + \begin{vmatrix} sz-1 & -sz \\ 2sz^2 & -sz^2 \end{vmatrix}}{\begin{vmatrix} sz-1 & sz \\ 2sz^2 & sz^2-1 \end{vmatrix}}$$

$$= \frac{2sz + sz^2 + s^2z^3}{1 - sz - sz^2 - s^2z^3} .$$

Before we leave this example we recall a result implied by Theorem 2 of Chapter III: If $H(x,y) = (1+xy)/(1-x-y+x^2y^2)$ generates $\{f(m,n)\}$, where $f(m,n)$ denotes the number of $(m+n)$-ominoes with m cells in the first row and n cells in the second row, then $H(x,x)-1$ generates the number of two row ν-ominoes, for $\nu = 1,2,\ldots$. Since $T(1,x)$ also generates the number of two row ν-ominoes, for $\nu = 1,2,\ldots,$ we must have $T(1,x) = H(x,x)-1$; the fact that this is indeed the case provides a check for our work.

If we want to obtain upper and lower bounds for the numbers $t(n)$ we use the fact that

$$(35) \qquad t_*(n) \leq t(n) \leq t^*(n) ,$$

which follows after summing on q in (19). It has already been observed that the sequences $\{t_*(n)\}$, $\{t(n)\}$, and $\{t^*(n)\}$ are generated by the rational functions $T_*(1,z)$, $T(1,z)$, and $T^*(1,z)$ respectively. If the sequences are increasing,

$$(36) \qquad \lim_{n \to \infty} (t_*(n))^{1/n} = \alpha , \quad \lim_{n \to \infty} (t(n))^{1/n} = \beta , \quad \text{and}$$

$$\lim_{n \to \infty} (t^*(n))^{1/n} = \gamma$$

all exist and are equal to the largest real roots of the auxilliary equations associated with $T_*(1,z)$, $T(1,z)$, and $T^*(1,z)$ respectively. For example, the equation associated with $T^*(1,z)$ is

$$(37) \qquad \det [a_{mn}] = 0 \quad ,$$

where

$$(38) \qquad a_{mn} = \begin{cases} f^*(m,m)-z^m , & \text{if } m = n \\ f^*(m,n) , & \text{if } m \neq n \end{cases} ;$$

the equation associated with $T_*(1,z)$ is defined by writing the asterisks in (38) as subscripts instead of superscripts.

If there are two or more objects with the same weight, the determinants used to calculate T_* or T^* are smaller than those used to find T . If relations (37) and (38) are used to find α and γ , it follows from (35) that for given ϵ , $\alpha > \epsilon > 0$,

(39) $$(\alpha-\epsilon)^n < t_*(n) \leq t(n) \leq t^*(n) < (\gamma+\epsilon)^n \quad ,$$

for all sufficiently large n. Furthermore, (39) provides sharp bounds for $t(n)$ if $f_*(m,n) = f^*(m,n)$, and $g_*(n) = g^*(n)$, for $m,n = 1,2,\ldots,$ since then $\alpha = \beta = \gamma$. Note also that if there is no more than one object with a given weight then $f_*(m,n) = f(m,n) = f^*(m,n)$, and $g_*(n) = p_n = g^*(n)$. Also, we might expect to obtain "good" bounds for $t(n)$ if there is only "slight" variation in the sets of numbers used to define $f^*(m,n)$ or $f_*(m,n)$ in (20) and (21).

The sequence $\{t(n)\}$ generated by $T(1,z)$ in (34) gives the number of two row n-ominoes for $n = 1,2,\ldots$. In this case α, β, and γ as defined in (36) are all equal to the largest real root of $z^3-z^2-z-1 = 0$ which is between 1.84 and 1.85. Thus, we can conclude that if $t(n)$ denotes the number of two row n-ominoes then

(40) $$(1.84)^n < t(n) < (1.85)^n \quad ,$$

for all sufficiently large n.

Read constructed the three row n-ominoes using the components illustrated in Figure 1(a) as the objects in his theorem; the label and object number of each component is also indicated in the figure. Unlike the two row case, the first k terms of a sequence of components corresponding to a three row n-omino need not form a connected set; such an n-omino is illustrated in Figure 1(b).

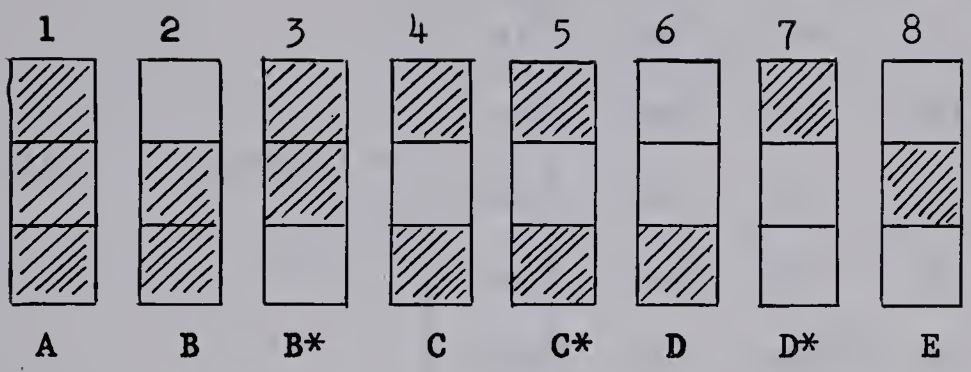


Figure 1(a)

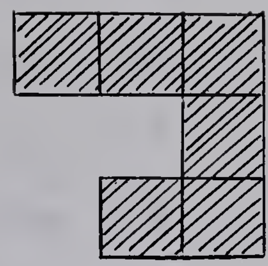


Figure 1(b)

The weight of a component is the number of cells it contains; we put $X = \{1,2,3,4,5,6,7,8\}$, $Y = X - \{4\}$, $Z = X - \{5\}$, $X_1 = X - \{5\}$, $X_2 = X - \{4,7\}$, $X_3 = X - \{4,6\}$, $X_4 = X - \{5,8\}$, $X_5 = \{1,5\}$, $X_6 = \{1,2,5,6\}$, $X_7 = \{1,3,5,7\}$, $X_8 = \{1,2,3,8\}$, and apply Read's Theorem to obtain a generating function for the number of $3 \times q$ n-ominoes:

(41) $$T(s,z) = sP[I - s\psi(z)]^{-1}\, s \quad ,$$

(42) $$P = \text{row vector } [1,1,1,1,0,1,1,1] \quad ,$$

(43) $$S = \text{column vector } [z^3,z^2,z^2,0,z^2,z,z,z] \quad ,$$

and

$$(44) \qquad I-s\psi(z) = \begin{bmatrix} 1-sz^3 & -sz^2 & -sz^2 & -sz^2 & 0 & -sz & -sz & -sz \\ -sz^3 & 1-sz^2 & -sz^2 & 0 & -sz^2 & -sz & 0 & -sz \\ -sz^3 & -sz^2 & 1-sz^2 & 0 & -sz^2 & 0 & -sz & -sz \\ -sz^3 & -sz^2 & -sz^2 & 1-sz^2 & 0 & -sz & -sz & 0 \\ -sz^3 & 0 & 0 & 0 & 1-sz^2 & 0 & 0 & 0 \\ -sz^3 & -sz^2 & 0 & 0 & -sz^2 & 1-sz & 0 & 0 \\ -sz^3 & 0 & -sz^2 & 0 & -sz^2 & 0 & 1-sz & 0 \\ -sz^3 & -sz^2 & -sz^2 & 0 & 0 & 0 & 0 & 1-sz \end{bmatrix}$$

In order to find β as defined in (36) we would have to find the largest real root of $\det [I - \psi(\frac{1}{z})] = 0$; instead of doing this, we will estimate β by means of (39) . We have under present assumptions

$$(45) \qquad [f^*(m,n)] = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 1 \\ 3 & 3 & 1 \end{bmatrix}, \quad \text{and} \quad [f_*(m,n)] = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 3 & 3 & 1 \end{bmatrix},$$

so that the auxilliary equations associated with $T^*(1,z)$ and $T_*(1,z)$ are $x^6-x^5-3x^4-2x^3-2x^2+1 = 0$, and $x^6-x^5-x^4-2x^3-2x^2-1 = 0$, respectively. The largest real root of the first of these equations is between 2.57 and 2.58 and the largest real root of the second is between 2.09 and 2.10 so that if $t(n)$ denotes the number of three row n-ominoes, then

$$(46) \qquad (2.09)^n < t(n) < (2.58)^n ,$$

for all sufficiently large n .

We conclude this chapter by finding difference equations for the number of three row n-ominoes, and at the same time illustrate another enumeration technique related to Read's Theorem.

If $\ell+m+n = \nu$, we again let $F_{\ell mn}$ denote the set of equivalence classes in S_ν containing ν-ominoes with ℓ, m, and n cells in the first, second, and third rows respectively; also, $f(\ell,m,n)$ is the number of elements in $F_{\ell mn}$. If V denotes a component illustrated in Figure 1(a), we let $V_{\ell mn}$ denote the subset of equivalence classes in $F_{\ell mn}$ containing ν-ominoes whose sequences of components end with V; also, $v(\ell,m,n)$ is the number of elements in $V_{\ell mn}$ and $V(x,y,z)$ is the generating function of the sequence $\{v(\ell,m,n): \ell,m,n = 0,1,2,\ldots\}$. We write $(V \cup W \cup \ldots)_{\ell mn}$ for $V_{\ell mn} \cup W_{\ell mn} \cup \ldots$, and $(v+w+\ldots)(\ell,m,n)$ for the number of elements in this set.

The component C* cannot end a sequence of components which form a ν-omino, so $C^*_{\ell mn}$ is empty; thus, we have by definition

(47) $$(f^*-a-b-b^*-c-d-d^*-e)(\ell,m,n) = 0 ,$$

for $\ell,m,n = 0,1,\ldots$, where $F^*_{\ell mn} = (A \cup B \cup \ldots)_{\ell mn}$. We will see presently that $F_{\ell mn}$ is contained in $F^*_{\ell mn}$.

Also, we have the following relations:

(48) $$b(\ell,m,n) - (f^*-d^*)(\ell-1,m-1,n) = 0 , \quad \ell,m > 1 , \quad n \geq 0 ,$$

(49) $$b^*(\ell,m,n) - (f^*-d)(\ell,m-1,n-1) = 0 , \quad \ell \geq 0 , \quad m,n > 1 ,$$

(50) $$c(\ell,m,n) - (a+c)(\ell-1,m,n-1) = 0 , \quad m \geq 0 , \quad \ell,n \geq 1 ,$$

(51) $$d(\ell,m,n) - (a+b+c+d)(\ell-1,m,n) = 0 , \quad m,n \geq 0 , \quad \ell > 1 ,$$

(52) $$d^*(\ell,m,n) - (a+b^*+c+d^*)(\ell,m,n-1) = 0 , \quad \ell,m \geq 0 , \quad n > 1 ,$$

(53) $$e(\ell,m,n) - (a+b+b^*+e)(\ell,m-1,n) = 0 , \quad \ell,n \geq 0 , \quad m > 1 .$$

To prove (48), note that if B is deleted from each ν-omino in $B_{\ell mn}$, $\ell,m > 1$, $n \geq 0$, then each of the elements of the equivalence classes in

(54) $$(A \cup B \cup B^* \cup C \cup D \cup E)_{\ell-1,m-1,n}$$

is obtained exactly once, but this is the set $(F^*-D^*)_{\ell-1,m-1,n}$. The existence of this one-one correspondence between the elements of $B_{\ell mn}$ and those of $(F^*-D^*)_{\ell-1,m-1,n}$ implies (48); the proofs of (49) - (53) are similar.

If A is deleted from each ν-omino in $A_{\ell mn}$ we obtain the elements of $F^*_{\ell-1,m-1,n-1}$ and some disconnected sets of cells. The ν-ominoes in $A_{\ell mn}$ which give rise to disconnected sets of cells may be characterized as sequences of components in which the last $k+2$ terms have one of the forms $(\ldots,B,C^*,\ldots,C^*,A)$, $(\ldots,B^*,C^*,\ldots,C^*,A)$,

$(\ldots,D,C^*,\ldots,C^*,A)$, or $(\ldots,D^*,C^*,\ldots,C^*,A)$, for $k = 1,2,\ldots$; we also have a sequence $(D,\ldots,D,C^*,\ldots,C^*,A)$ if $\ell > n$, $(C^*,\ldots,C^*,A)$ if $\ell = n$, and $(D^*,\ldots,D^*,C^*,\ldots,C^*,A)$ if $\ell < n$. The number of elements in $A_{\ell mn}$ is therefore,

$$(55) \qquad a(\ell,m,n) = 1+f^*(\ell-1,m-1,n-1) + \sum_{k=2}^{\infty}(b+b^*+d+d^*)(\ell-k,m-1,n-k) .$$

Using the last expression for $a(\ell,m,n)$, we find the difference $a(\ell,m,n) - a(\ell-1,m,n-1)$, and obtain after transposing a term,

$$(56) \qquad \begin{aligned} a(\ell,m,n) &= a(\ell-1,m,n-1) + f^*(\ell-1,m-1,n-1) \\ &+ (b+b^*+d+d^*-f^*)(\ell-2,m-1,n-2) \\ &= 2(\ell-1,m,n-1) + f^*(\ell-1,m-1,n-1) \\ &- (a+c+e)(\ell-2,m-1,n-2) , \end{aligned}$$

for $\ell,n \geq 2$, $m \geq 1$.

Each element of $B_{\ell mn}$ or $D_{\ell mn}$ can be reflected about its midsection to obtain elements of $B^*_{nm\ell}$ or $D^*_{nm\ell}$ respectively, and **vice versa**; from this it follows that

$$(57) \qquad b(\ell,m,n) = b^*(n,m,\ell), \quad \text{and} \quad d(\ell,m,n) = d^*(n,m,\ell) .$$

The difference equations in (48) - (53), (56) and (57)

provide a means of calculating $f^*(\ell,m,n)$ without using generating series. The sets F^* differ from the sets F in that ν-ominoes in F have cells in exactly three rows, but this is not necessarily the case for ν-ominoes in F^*. These difference equations imply relations between the generating series:

(58) $$(F^*-A-B-B^*-C-D-D^*-E)(x,y,z) = 0 ,$$

(59) $$(B-xyF^*+xyD)(x,y,z) = xy ,$$

(60) $$(B^*-yzF^*+yzD^*)(x,y,z) = yz ,$$

(61) $$(C-xzA-xzC)(x,y,z) = xz ,$$

(62) $$(D-xA-xB-xC-xD)(x,y,z) = x ,$$

(63) $$(D^*-zA-zB^*-zC-zD^*)(x,y,z) = z ,$$

(64) $$(E-yA-yB-yB^*-yE)(x,y,z) = y ,$$

(65) $$(A-xzA+x^2yz^2A-xyzF^*+x^2yz^2C+x^2yz^2E)(x,y,z) = xyz ,$$

(66) $$B(x,y,z) = B^*(z,y,x) ,$$

(67) $$D(x,y,z) = D^*(z,y,x) .$$

When $x = y = z$, this system reduces to a system of six equations linear in the functions F^*, A, $B = B^*$, C, $D = D^*$, and E . Applying Cramer's rule to this system, we are led at once to a determinant of a matrix having the form of (44) .

CHAPTER V

PLANE TREES

In the proof of Lemma 3 in Chapter II we described Eden's method for assigning a sequence of binary digits, $W(X)$, to each n-omino, X. If $W(X) = (\beta_1, \beta_2, \ldots, \beta_{3n-1})$, we must have $\beta_{3n-3} = \beta_{3n-2} = \beta_{3n-1} = 0$, since C_n cannot have any directed edges going from it to other cells in X. Thus, the sequence $(0, \beta_1, \beta_2, \ldots, \beta_{3n-4}) = (\alpha_1, \alpha_2, \ldots, \alpha_{3n-3})$ is such that

$$(1) \qquad \sum_{i=1}^{3j} \alpha_i \geq j \quad ,$$

for $j = 1,2,\ldots,n-1$, since $(0, \beta_1, \beta_2, \ldots, \beta_{3j}, 0,0,0,\ldots)$ corresponds to an r-omino $(r \geq j)$ which forms a part of X. We will show presently that there is a natural one-one correspondence between binary sequences $(\alpha_1, \alpha_2, \ldots, \alpha_{3n-3})$ which satisfy (1), and the elements of a class of trees embedded in the Euclidean plane having nodes with valency 1 or 4. Besides exploring this idea further, we will show that various kinds of plane trees are enumerated by sums of the type studied in Chapter I.

Harary, Prins, and Tutte have written joint and separate papers

on the subject of plane trees [6], [7], [23]; for completeness we reproduce their definitions here. A <u>tree</u> is a connected graph which contains no polygons, and a <u>plane tree</u> is a tree embedded in the plane with no distinct edges intersecting. A <u>rooted</u> (plane) tree is a (plane) tree in which one node is distinguished as a <u>root</u>. A <u>planted</u> plane tree is a rooted plane tree in which the root has valence 1. Two trees are <u>isomorphic</u> if there is a one-one mapping of the nodes of one onto the nodes of the other which preserves adjacency. Two plane trees are <u>map-isomorphic</u> if there exists an orientation preserving homeomorphism of the plane onto itself which maps one onto the other. Similarly, two rooted (plane) trees are (map-) isomorphic if there exists a (map-) isomorphism between them in which the roots correspond.

We let P_n denote the set of non-map-isomorphic, planted, plane trees with n nodes, and let $p(n)$ denote the number of elements in P_n. The <u>level</u> of a node in an element of P_{n+2} is the length of the path from the root to that node. Corresponding to each composition $(1,1,a_2,a_3,\ldots)$ of $n+2$ into an unrestricted number of positive parts, there is a subset of P_{n+2} containing elements which have exactly a_i nodes with level i, for $i = 2,3,\ldots$. The nodes having the same level can be arranged in rows so that no edges in the tree cross, and the nodes in each row can be labelled consecutively from left to right in successive rows beginning with the root of the tree.

Evidently, all of the edges of a tree in P_{n+2} occur between consecutive levels such that each node in the j^{th} level is connected

to exactly one node in the $(j-1)^{st}$ level, and every node in the $(j-1)^{st}$ level is connected to a (possibly empty) set of consecutively labelled nodes in the j^{th} level. If a_{j-1} and a_j denote the number of nodes in the $(j-1)^{st}$ and j^{th} levels respectively, then the number of ways edges may be drawn between these levels is $\binom{a_{j-1} + a_j - 1}{a_j}$, the number of ways of selecting a_j objects from a set of a_{j-1} objects with repetitions of a choice permitted. Thus, the number of elements of P_{n+2} corresponding to the composition $(1,1,a_2,a_3,\ldots)$ of $n+2$ is

$$(2) \qquad \binom{1+a_2-1}{a_2}\binom{a_2+a_3-1}{a_3}\cdots,$$

and hence

$$(3) \qquad p(n+2) = \sum \binom{a_1+a_2-1}{a_2}\binom{a_2+a_3-1}{a_3}\cdots,$$

where the sum extends over all compositions $(a_1,a_2,\ldots,a_i)$ of n into an unrestricted number of positive parts, and the composition with one part contributes 1 to the sum.

In order to be consistent with the defintions and notation of Chapter I we write $b(n)$ for $p(n+2)$; then the numbers $b(a,n)$, $b_k(n)$, ... have obvious combinatorial interpretations. Using $f(m,n) = \binom{m+n-1}{n}$, we have

(4) $$F(x,y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \binom{m+n-1}{n} x^m y^n = \frac{xy}{(1-x)(1-x-y)} ,$$

and substituting this function in (1.15) gives

(5) $$B_{k+1}(x,y) = \frac{xy}{1-xy} \cdot B_k(x, \frac{1}{1-xy}) ,$$

for $k = 1,2,\ldots$. We have $B_1(x,y) = xy/(1-xy)$; this fact along with (5) and an induction argument gives

(6) $$B_k(x,y) = \frac{x^k y}{\{A_{k-1}(x)-xyA_{k-2}(x)\}\ \{A_k(x)-xyA_{k-1}(x)\}} ,$$

where $A_0(x) = A_1(x) = 1$, and $A_{k+2}(x) = A_{k+1}(x)-xA_k(x)$, for $k = 0,1,\ldots$. The difference equation satisfied by $\{A_k(x)\}$ can be used to prove that

(7) $$B_1(x,y)+\ldots+B_k(x,y) = \frac{xyA_{k-1}(x)}{A_k(x)-xyA_{k-1}(x)} .$$

Since $B_1+B_2+\ldots = B$, we put $y = 1$ in (7) and use the difference equation satified by $\{A_k\}$ to find

(8) $$B(x,1) = \lim_{k \to \infty} \frac{xA_{k-1}(x)}{A_{k+1}(x)} = \lim_{k \to \infty} \frac{A_k(x)}{A_{k+1}(x)} - 1 .$$

Using the fact that the limit of a product is the product of the limits, (8) implies

$$(9) \qquad \{1+B(x,1)\}^2 = \lim_{k\to\infty} \frac{A_k(x)}{A_{k+1}(x)} \cdot \frac{A_{k-1}(x)}{A_k(x)} = \frac{B(x,1)}{x} .$$

Solving the quadratic equation in (9) for $B(x,1)$ and recalling that $b(n) > 0$, we obtain

$$(10) \qquad B(x,1) = \frac{1-2x-\sqrt{1-4x}}{2x} = \sum_n \frac{1}{n+1}\binom{2n}{n} x^n ;$$

thus, $p(n+2) = \frac{1}{n+1}\binom{2n}{n}$. This generating function was also found in [6] by applying Pólya's fundamental theorem.

To find $b(a,n)$, the number of elements of P_{n+2} with exactly a nodes in the second level we use (7), (8), and (10) and obtain

$$(11) \qquad \begin{aligned} B(x,y) &= \lim_{k\to\infty} xy \left\{ \frac{A_k(x)}{A_{k-1}(x)} - xy \right\}^{-1} \\ &= \frac{xy[1+B(x,1)]}{1-xy[1+B(x,1)]} = \frac{\frac{1}{2}y(1-\sqrt{1-4x})}{1-\frac{1}{2}y(1-\sqrt{1-4x})} . \end{aligned}$$

By definition of $B(x,y)$, $b(a,n)$ is the coefficient of $y^a x^n$ in the expansion of the last member of (11); thus,

$$(12) \qquad \left\{ \frac{1-\sqrt{1-4x}}{2} \right\}^a = \sum_{n=1}^{\infty} b(a,n)x^n .$$

The number of elements of P_{n+2} with exactly a nodes in the second level and exactly $k+2$ levels is $b_k(a,n)$, which is

generated by $B_k(x,y)$ given in (6). Similarly, the number of elements P_{n+2} with an arbitrary number of nodes in the second level and exactly $k+2$ levels is $b_k(n)$, generated by $B_k(x,1)$.

The boundary conditions and difference equation satisfied by $\{A_k\}$ imply

(13) $$A_k(x) = (\theta_1^{k+1} - \theta_2^{k+1})/(\theta_1 - \theta_2) \quad ,$$

where $(1-\theta_1 z)(1-\theta_2 z) = 1-z+xz^2$. Thus, for example, from (6) and (13) we find the explicit formula

(14) $$B_k(x,1) = \frac{x^k}{A_k(x)A_{k+1}(x)}$$

$$= \frac{x^k(1-4x)}{(\theta_1^{2k+3} + \theta_2^{2k+3} - x^{k+1})} \quad .$$

The first equality in (14) and the difference equation satisfied by $\{A_k\}$ may be used effectively to calculate

(15) $$B_1(x,1) = \frac{x}{1-x}, \qquad B_2(x,1) = \frac{x^2}{(1-x)(1-2x)}$$

$$B_3(x,1) = \frac{x^3}{(1-2x)(1-3x+x^2)}, \qquad B_4(x,1) = \frac{x^4}{(1-3x+x^2)(1-4x+3x^2)} \quad .$$

Of course, (14) also implies that each of the sequences $\{b_k(n) : n = 1,2,\ldots\}$ satisfies a linear, homogeneous difference equation of order k with constant coefficients.

If all of the nodes of a tree have valence either 1 or $k+1$, then the tree has $kn+2$ nodes altogether. This is because the tree with two nodes has this property and every tree of this kind with more than two nodes can be drawn by joining k nodes at a time to nodes of valence 1 in a tree already possessing this property. We let P^k_{kn+2} denote the subset of P_{kn+2} containing trees with nodes having valence either 1 or $k+1$, and let $p_k(kn+2)$ denote the number of elements in this set.

Each element X of P^k_{kn+2} is such that each level of nodes except the first level contains a multiple of k nodes. If there are ka_{j-1} and ka_j nodes in the $(j-1)^{st}$ and j^{th} levels of X, the nodes in the j^{th} level may be joined to those in the $(j-1)^{st}$ level in any of $\binom{ka_{j-1}}{a_j}$ ways. This follows since each block of k nodes in the j^{th} level must be joined to exactly one node in the $(j-1)^{st}$ level, and each node in the $(j-1)^{st}$ level is joined to at most k nodes in the j^{th} level. Since there are exactly k nodes in the second level of every element of P^k_{kn+2}, for $n > 0$, the number of elements in P^k_{kn+2} is $b(1,n) = p_k(kn+2)$, where

$$b(n) = \sum \binom{ka_1}{a_2}\binom{ka_2}{a_3}\cdots\binom{ka_{i-1}}{a_i}, \tag{16}$$

and the sum extends over all compositions of n into an unrestricted number of positive parts.

Using $f(m,n) = \binom{km}{n}$, we have

$$(17) \qquad F(x,y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \binom{km}{n} x^m y^n = \frac{x(1+y)^k}{1-x(1+y)^k} - \frac{x}{1-x} ,$$

so that (1.17) becomes

$$(18) \qquad B(x,y) = \frac{xy}{1-xy} + \frac{1}{2\pi i} \int_c \frac{xy(1+s)^k}{s^k-xy(1+s)^k} B(x,s) \frac{ds}{s} .$$

It is also easy to prove that

$$(19) \qquad B(x,y) = \frac{y\partial B(x,0)}{1-y\partial B(x,0)} ,$$

where $\partial^i B(x,0)$ denotes the i^{th} partial derivative with respect to y of $B(x,y)$ at $y = 0$. This follows from the fact that

$$(20) \qquad b(i,n) = \sum b(1,a_1)b(1,a_2)\ldots b(1,a_i) ,$$

where the sum in (20) extends over all compositions $(a_1,a_2,\ldots,a_i)$ of n into exactly i positive parts.

Substituting the expression for $B(x,y)$ given by (19) into the right and left members of (18) and then differentiating with respect to y at $y = 0$ in the resulting equation gives

$$\text{(21)} \qquad \partial B(x,0) = x + \frac{1}{2\pi i}\int_c x \left(\frac{s+1}{s}\right)^k \frac{\partial B(x,0)}{1-\partial B(x,0)}\, ds$$

$$= x\,\{1+\partial B(x,0)\}^k .$$

A purely combinatorial proof of (21) follows if it is noted that each element of P^k_{kn+2} can be interpreted as a k-tuple $(X_1,X_2,..,X_k)$ with $X_i \in P^k_{ka_i+2}$, for $i = 1,2,...,k$, and where $(a_1,a_2,...,a_k)$ is a composition of n into non-negative parts.

Solving (21) for $\partial B(x,0)$ involves the problem of finding $y = \partial B(x,0) + 1$ as a power series in x such that $xy^k-y+1 = 0$; essentially the same problem was solved by Kelly [11] who used a method for inverting trinomials due to Frame [2]. Following this procedure we obtain

$$\text{(22)} \qquad \partial B(x,0) = \sum_{n=1}^{\infty} \frac{1}{(k-1)n+1} \binom{kn}{n} x^n ,$$

so that the number of non-map-isomorphic planted plane trees having nodes with valence either 1 or k+1 is

$$\text{(23)} \qquad p_k(kn+2) = \frac{1}{(k-1)n+1} \binom{kn}{n}, \qquad n = 0,1, \ldots .$$

Frame's result is as follows: Suppose we have a non-associative multiplication defined on $\{a_1,a_2,...,a_n\}$; then the number of different products which arise if factors of $a_1a_2...a_n$ are taken r+1 at a time

by inserting parentheses and keeping the order of the a_i unchanged is $\frac{1}{kr+1}\binom{(r+1)k}{k}$, if $n = kr+1$, and is zero otherwise.

From (10) and (23) we see that $T_{2n+2} = P^2_{2n+2}$, the set of non-map-isomorphic planted plane trees having nodes with valence either 1 or 3 , has the same number of elements as P_{n+2} . Sabidussi [20], in his review of the paper of Harary, Prins, and Tutte [6] , remarks that the natural one-one correspondence which they establish between the elements of these sets is the most important result in their paper. We are going to establish a simple one-one correspondence between the elements of either of these sets and the elements of the set S^2_{2n-2} of all binary sequences $(a_1, a_2, \ldots, a_{2n-2})$ containing exactly $n-1$ units such that

$$(24) \qquad \sum_{i=1}^{2j} a_i \geq j , \quad \text{for} \quad j = 1,2,\ldots,n-1 .$$

The one-one correspondence induced in this way between the elements of T_{2n+2} and P_{n+2} is much simpler than the one given in [6].

We suppose the elements of P_{n+2} and T_{2n+2} to be labelled in the way described in the third paragraph of this chapter. A unique binary sequence of length $2n+3$ can be assigned to each element X of P_{n+2} as follows: The sequence starts $(1,0,\ldots)$ to indicate that the root of X has exactly one edge going from it to the node in the second row. Next, we add as many 1's to the sequence as there are edges going from node i to the nodes in the row above it and

then add 0 to the sequence, for $i = 2,3,\ldots,n+2$, respectively. There are as many 1's and o's in this sequence as there are edges and nodes in the tree respectively; hence, the sequence has $(n+1)+(n+2) = 2n+3$ digits. Since, for $n > 2$, each sequence begins and ends in the same way, namely $(1,0,1,\ldots,0,0)$, we can delete five **digits** of this sequence to obtain $W(X) = (a_1,a_2,\ldots,a_{2n-2})$. It must be that $W(X)$ satisfies (24), since $(a_1,a_2,\ldots,a_{2j})$ is the set of digits which begins $W(Y)$, where Y is a part of X with $j+2$ or more edges. Furthermore, if $(a_1,a_2,\ldots,a_{2n-2})$ satisfies (24) we can construct an element of P_{n+2} using $(1,0,1,a_1,\ldots,a_{2n-2},0,0)$.

Suppose $X \in T_{2n+2}$ and that X is labelled in the same way as any other elements of P_{2n+2}. Let $i_1 < i_2 < \ldots$ denote the nodes of X having valence 3 and set $i_j = j*$. Corresponding uniquely to X is the sequence $(\alpha_1,\beta_1)(\alpha_2,\beta_2)\ldots(\alpha_n,\beta_n)$, where the digits α_j and β_j correspond to the nodes joined to $j*$ on the left and right respectively in the level above $j*$. A digit in this sequence is 0 or 1 if the node to which it corresponds has valence 1 or 3 respeec-tively. It is easy to see that every element of T_{2n+2} has exactly n nodes of valence 3; also, it is clear that $n*$ is such that $\alpha_n = \beta_n = 0$. Thus, we can delete the last two digits of $(\alpha_1,\beta_1)\ldots(\alpha_n,\beta_n)$ without loosing any information about X; hence, we define $W*(X) = (\alpha_1,\beta_1,\alpha_2,\beta_2,\ldots\alpha_{n-1},\beta_{n-1}) = (a_1,a_2,\ldots,a_{2n-2})$. Again $W*(X)$ must satisfy (24) because a part of $W*(X)$ corresponds to a smaller part of X having the same properties. Furthermore, it is easy to check that if $(a_1,a_2,\ldots,a_{2n-2})$ is an elment of S^2_{2n-2}, then we can construct an element of T_{2n+2}

using $(a_1, a_2, \ldots, a_{2n-2}, 0, 0)$. In Figure 1, the correspondence between the elements of S^2_{2n-2}, P_{n+2} , and T_{2n+2} is shown for $n = 3$.

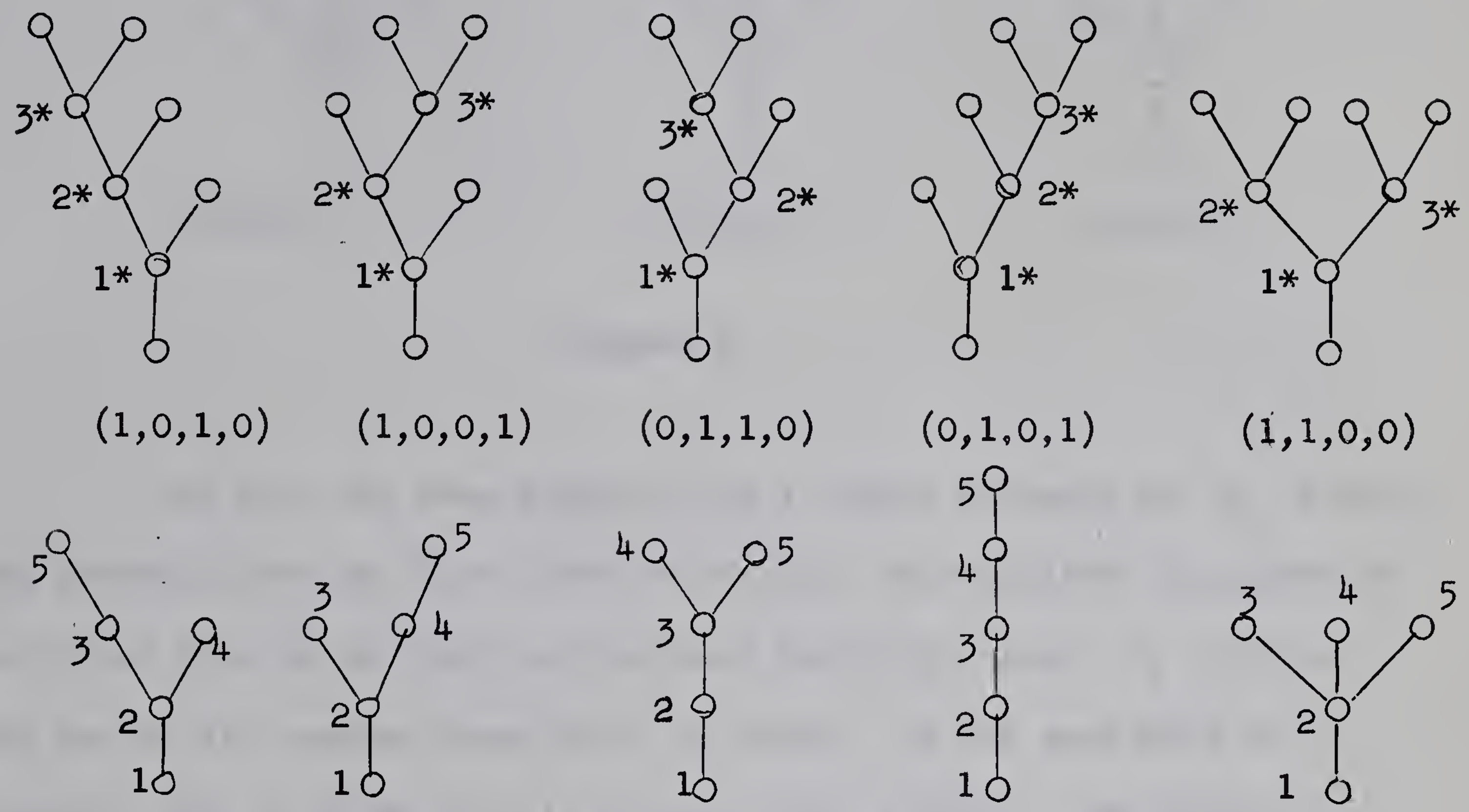


Figure 1.

The correspondence between the elements of S^2_{2n-2} and T_{2n+2} is a special case of a more general result. Let S^k_{kn-k} denote the set of binary sequences $(a_1, a_2, \ldots, a_{kn-k})$ containing exactly $n-1$ units such that

$$\sum_{i=1}^{kj} a_i \geq j , \qquad \text{for} \qquad j = 1,2,\ldots,n-1 . \tag{25}$$

There exists a natural one-one correspondence between the elements of S^k_{kn-k} and P^k_{kn+2} . The proof of this fact is a straightforward generalization of the argument presented in the last paragraph. This correspondence for $k = 3$ and $n = 2$ is illustrated in Figure 2.

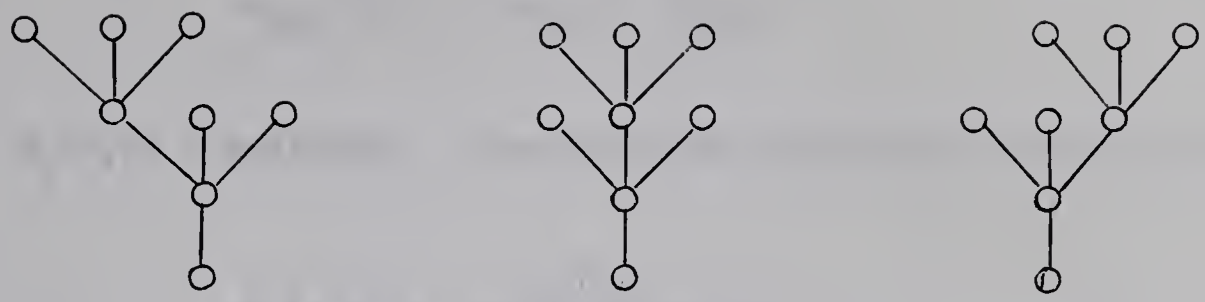


Figure 2.

We give the name <u>cypress tree</u> to those elements of P_n having the property that the first node on the left in each level is joined to the first node on the left in the level above it; also, C_n denotes the set of all cypress trees with n nodes. By the same kind of argument used to prove (3), it follows that $c(n+2)$, the number of cypress trees with $n+2$ nodes, is equal to

$$(26) \qquad b(n) = \sum \binom{a_1+a_2-2}{a_2-1}\binom{a_2+a_3-2}{a_3-1}\cdots\binom{a_{i-1}+a_i-2}{a_i-1} ,$$

where the sum extends over all compositions $(a_1,a_2,\ldots,a_i)$ of n into an unrestricted number of positive parts.

Since $f(m,n) = \binom{m+n-2}{n-1}$, we have

$$(27) \qquad F(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\binom{m+n-2}{n-1} x^m y^n = \frac{xy}{1-x-y} ;$$

hence, substituting this into (1.15) gives

(28) $$B_{k+1}(x,y) = xyB_k(x, \frac{1}{1-xy}) \ .$$

We have $B_1(x,y) = xy/(1-xy)$, and a simple induction using (28) gives

(29) $$B_k(x,y) = \frac{x^k y}{A_k(x)-xyA_{k-1}(x)} \ ,$$

where the A_k's are the polynomials defined after (6). From (29) it is evident that $\{b_k(n)\}$, where $b_k(n)$ denotes the number of cypress trees with exactly $k+2$ levels and $n+2$ nodes, is generated by $B_k(x,1) = x^k/A_{k+1}(x)$, a rational function. For example, when $k = 1,2,3,4,$ and 5 , $B_k(x,1)$ is respectively

(30) $$\frac{x}{1-x} \ , \quad \frac{x^2}{1-2x} \ , \quad \frac{x^3}{1-3x+x^2} \ , \quad \frac{x^4}{1-4x+3x^2} \ , \quad \text{and} \quad \frac{x^5}{1-5x+6x^2-x^3} \ .$$

The third of these relations implies $b_3(n) = F_{2n}$ where $\{F_i\}$ denotes the Fibonacci sequence; thus, the number of cypress trees with exactly five levels and $n+2$ nodes is F_{2n}, for $n = 1,2,\ldots$.

Suppose α and β are real numbers such that $t^2-\alpha t+\beta = 0$ has complex roots; then it is known that the Taylor expansion of $(1-\alpha z+\beta z^2)^{-1}$ has infinitely many negative coefficients. Now since

(31) $$\frac{1}{1-z+xz^2} = \sum_{k=o}^{\infty} A_k(x)z^k \ ,$$

we must have $A_k(x) \leq 0$ for infinitely many k whenever $x = 1/(4-\epsilon)$,

$4 > \epsilon > 0$. But $x^{[\frac{k}{2}]} A_k(\frac{1}{x}) = x^{[\frac{k}{2}]} + \dots$, so for some k , $A_k(\frac{1}{x}) = 0$ has a real root arbitrarily close to, but less than 4 . This imples $b_k(n) > (4-\epsilon)^n$, for sufficiently large k and n .

CHAPTER VI

SOME APPLICATIONS TO THE THEORY OF NUMBERS

It is sometimes of interest in the Theory of Numbers to evaluate or estimate sums which extend over all or only a special subset of the compositions of n. Perhaps the most common sum of this type is

$$r(n) = \sum_{d|n} s(d) , \tag{1}$$

where the sum extends over all compositions of n with the parts equal; that is, the sum extends over all positive divisors of n. The formulas of Chapter I can be specialized in order to deduce some of the well known facts about generating series of sequences $\{r(n)\}$ defined as in (1).

Suppose $W(x)$ and $F(x)$ generate $\{w(n)\}$ and $\{f(n)\}$ respectively; then define $F(x,y)$ and $G(x)$ in (1.7) by

$$f(m,n) = \begin{cases} f(n), & \text{if } m = n \\ 0 , & \text{otherwise} \end{cases} , \text{ and} \tag{2}$$

$$g(n) = f(n)\, w(n) . \tag{3}$$

For any composition $(a_1, a_2, \ldots, a_i)$ of n we have

$$(4) \qquad f(a_1,a_2)\, f(a_2,a_3) \cdots f(a_{i-1},a_i)\, g(a_i) = \begin{cases} w(a)f^{n/a}(a), & \text{if } a\,|\,n \\ 0\,, & \text{otherwise} \end{cases},$$

so that

$$(5) \qquad b(a,n) = \begin{cases} w(a)f^{n/a}(a), & \text{if } a\,|\,n \\ 0\,, & \text{otherwise} \end{cases}, \quad \text{and}$$

$$(6) \qquad b(n) = \sum_{d\,|\,n} w(d)f^{n/d}(d) \quad .$$

The definitions in (2) and (3) also imply $F(x,y) = F(xy)$, and

$$(7) \qquad G(x) = \sum_{n=1}^{\infty} f(n)w(n)x^n = \frac{1}{2\pi i} \int_c F(xs)W(\tfrac{1}{s})\, \frac{ds}{s} \quad ,$$

where c is a contour in the s plane which includes the singularities of $W(\frac{1}{s})/s$, but excludes those of $F(xs)$.

The case when $f(n) = 1$, for $n = 1,2,\ldots,$ is of particular interest. Here $F(x) = x/(1-x)$ and $G(x) = W(x)$; thus, (1.13) implies $B_1(x,y) = W(xy)$, and (1.15) implies

$$(8) \qquad B_{k+1}(x,y) = \frac{1}{2\pi i} \int_c \frac{xyB_k(x,s)}{s(s-xy)}\, ds = B_k(x,xy) \quad .$$

From (8) we deduce that $B_k(x,y) = W(x^k y)$, and hence that

$$(9) \qquad B(x,y) = W(xy) + W(x^2y) + W(x^3y) + \cdots \quad .$$

Under present assumptions, the functions $B^j(x,y)$ are easily found since the system (1.23) becomes

$$(10) \qquad \frac{\partial^p B^j}{p!} = w(p)x^p + \frac{\partial^p B^j}{p!} x^p ,$$

for $p = 1,2,\ldots,j$, so that

$$(11) \qquad \frac{\partial^p B^j}{p!} = \frac{w(p)x^p}{(1-x^p)} .$$

From (11) we deduce that

$$(12) \qquad B^j(x,y) = \sum_{p=1}^{j} \frac{w(p)x^p y^p}{(1-x^p)} ,$$

and letting j tend to infinity in (12) we obtain

$$(13) \qquad B(x,y) = \sum_{p=1}^{\infty} \frac{w(p)x^p y^p}{(1-x^p)} .$$

Combining (5), (9), and (13), the final result may be written

$$(14) \qquad B(x,y) = \sum_{n=1}^{\infty} \sum_{d|n} w(d)y^d x^n$$

$$= \sum_{p=1}^{\infty} \frac{w(p)x^p y^p}{(1-x^p)} = \sum_{n=1}^{\infty} W(x^n y) .$$

From (14) we can deduce many well known results (see pages 257-258 of Hardy and Wright [9]). For example, if $w(n) = \phi(n)$, the Euler ϕ-function, it is known that $\sum_{d|n} \phi(d) = n$; hence, (14) implies

(15) $$\sum_{n=1}^{\infty} \sum_{d|n} \phi(d) y^d x^n = \sum_{n=1}^{\infty} \frac{\phi(n) x^n y^n}{(1-x^n)} \quad , \quad \text{and}$$

(16) $$\frac{x}{(1-x)^2} = \sum_{n=1}^{\infty} \sum_{d|n} \phi(d) x^n = \sum_{n=1}^{\infty} \frac{\phi(n) x^n}{(1-x^n)} \quad .$$

Also, if $w(n) = \mu(n)$, the Möbius function, it is known that

(17) $$\sum_{d|n} \mu(d) = \begin{cases} 1 , & \text{if } n = 1 \\ 0 , & \text{otherwise} \end{cases} \quad ;$$

thus, (14) implies

(18) $$\sum_{n=1}^{\infty} \sum_{d|n} \mu(d) y^d x^n = \sum_{n=1}^{\infty} \frac{\mu(n) x^n y^n}{(1-x^n)} \quad \text{and}$$

(19) $$x = \sum_{n=1}^{\infty} \sum_{d|n} \mu(d) x^n = \sum_{n=1}^{\infty} \frac{\mu(n) x^n}{(1-x^n)} \quad .$$

Finally, if $w(n) = n^k$, $\sum_{d|n} d^k = \sigma_k(n)$; usually, we write

$\sigma_0 = \tau$, $\sigma_1 = \sigma$. For $k = 0,1,2,$ the generating function of $\{n^k\}$ is $x(1-x)^{-1}$, $x(1-x)^{-2}$, and $(x+x^2)(1-x)^{-3}$ respectively; hence, (14) implies

$$\sum_{n=1}^{\infty} \tau(n)x^n = \sum_{n=1}^{\infty} \frac{x^n}{(1-x^n)} , \tag{20}$$

$$\sum_{n=1}^{\infty} \sigma(n)x^n = \sum_{n=1}^{\infty} \frac{nx^n}{(1-x^n)} = \sum_{n=1}^{\infty} \frac{x^n}{(1-x)^2} , \tag{21}$$

$$\sum_{n=1}^{\infty} \sigma_2(n)x^n = \sum_{n=1}^{\infty} \frac{n^2x^n}{(1-x^n)} = \sum_{n=1}^{\infty} \frac{x^n+x^{2n}}{(1-x^n)^3} . \tag{22}$$

Another class of problems in the theory of numbers asks for the number of compositions of n which satisfy certain requirements. Some of these problems stipulate how many parts may be used and which numbers may serve as parts of the composition; a generating function for compositions of this kind is easily obtained. Let $b(n)$ denote the number of compositions of n which satisfy these stipulations and define

$$p(n) = \begin{cases} 1 , & \text{if a composition may have } n \text{ parts} \\ 0 , & \text{otherwise} \end{cases} , \tag{23}$$

$$w(n) = \begin{cases} 1 , & \text{if } n \text{ may serve as a part in a composition} \\ 0 , & \text{otherwise} \end{cases}$$

Suppose $P(x)$, $W(x)$, and $B(x)$ generate $\{p(n)\}$, $\{w(n)\}$, and $\{b(n)\}$ respectively, then it is easy to show that $B(x) = P(W(x))$ (for a proof, see Klarner [13]; this result is a generalization of a theorem proved by Moser and Whitney [14]).

Problems involving enumeration of compositions become more difficult when in addition to stipulating the number and type of parts, the pairs of consecutive parts are restricted in some way as well. A common type of composition satisfying properties of this kind is a partition of n which we will consider presently.

Suppose a set S of positive parts and a subset P of $S \times S$ are given and define

$$\text{(24)} \qquad f(m,n) = \begin{cases} w(m), & \text{if } (m,n) \in P \\ 0, & \text{otherwise} \end{cases},$$

$$\text{(25)} \qquad g(n) = \begin{cases} w(n), & \text{if } n \in S \\ 0, & \text{otherwise} \end{cases},$$

where $w(1), w(2), \ldots$ are the <u>weights</u> of $1,2,\ldots$.

Now defining $b(n)$ in the usual way we have

$$\text{(26)} \qquad b(n) = \sum f(a_1,a_2)\, f(a_2,a_3) \ldots f(a_{i-1},a_i)\, g(a_i)$$

$$= \sum w(a_1)\, w(a_2) \ldots w(a_i) ,$$

where the first sum extends over all compositions of n and the second sum extends only over those compositions $(a_1, a_2, \dots, a_i)$ of n such that $a_j \in S$ and $(a_{j-1}, a_j) \in P$, for $j = 1,2,\dots,i$. Thus, the general problem we have set out to solve may be treated using the methods of Chapter I.

For example; to find the generating function of $\{b(n)\}$, where $b(n)$ denotes the number of ways of expressing n as a sum of one or more consecutive numbers, we put $g(n) = 1$, and

$$(27) \qquad f(m,n) = \begin{cases} 1, & \text{if } m = n+1 \\ 0, & \text{otherwise} \end{cases},$$

so that $G(x) = x/(1-x)$, $F(x,y) = x^2y/(1-xy)$ and

$$(28) \qquad b(a,n) = \begin{cases} 1, & \text{if } n-a = 1+2+\dots+i, \text{ for } i \geq 0 \\ 0, & \text{otherwise} \end{cases}.$$

Using (1.15) we get

$$(29) \qquad B_{k+1}(x,y) = \frac{1}{2\pi i} \int_c \frac{x^2y^2B_k(x,s)}{s(s-xy)}\, ds = xyB_k(x,xy),$$

so that evidently $B_k(x,y) = x^{k(k+1)/2}\, y^k/(1-x^ky)$, and hence

$$(30) \qquad B(x,y) = \sum_{k=1}^{\infty} \frac{x^{k(k+1)/2}\, y^k}{1-x^ky}.$$

To find the generating function of $\{b(n)\}$, where $b(n)$ denotes the number of compositions of n with contiguous parts distinct, we set $g(n) = 1$, and define

$$(31) \qquad f(m,n) = \begin{cases} 1, & \text{if } m \neq n \\ 0, & \text{otherwise} \end{cases} ;$$

in this case we have $G(x) = x/(1-x)$ and $F(x,y) = xy(x+y-2xy)/(1-x)(1-y)(1-xy)$. Substituting these functions into (1.17), integrating, and transposing a term gives

$$(32) \qquad B(x,xy) + B(x,y) = \{1+B(x,1)\} \frac{xy}{1-xy} .$$

Now we write $x^k y$ for y in (32), multiply the resulting identity by $(-1)^k$, for $k = 0,1,\ldots,K$, and sum over k to obtain

$$(33) \qquad B(x,y) + (-1)^K B(x,x^{K+1}y) = \{1+B(x,1)\} \sum_{k=0}^{K} \frac{(-1)^k x^{k+1} y}{1-x^{k+1} y} .$$

But $B(x,x^{K+1}y)$ tends to zero as K tends to infinity for $|x|$ sufficiently small; thus (33) implies

$$(34) \qquad \frac{B(x,1)}{1+B(x,1)} = \sum_{k=1}^{\infty} \frac{(-1)^{k+1} x^k}{1-x^k} \equiv T(x) .$$

Using (23), we see that $B/(1+B)$ generates

$$(35) \qquad \sum (-1)^i \, b(a_1) \, b(a_2) \ldots b(a_i) ,$$

where the sum extends over all compositions of n ; also, (14) implies $T(x)$ generates

(36) $$\sum_{d|n} (-1)^{d+1} = \tau_o(n) - \tau_e(n) = \tau^*(n) ,$$

where $\tau_o(n)$ and $\tau_e(n)$ denote the number of odd divisors and the number of even divisors of n respectively. Thus, if 2^k is the highest power of 2 which divides n , $\tau^*(n) = (1-k)\ \tau(n/2k)$.

The relation (34) also implies $B(x,1) = T(x)/(1-T(x)$, so that using (23) again, we have

(37) $$b(n) = \sum \tau^*(a_1)\ \tau^*(a_2) \ldots \tau^*(a_i) ,$$

where the sum extends over all compositions of n . Equating (35) and (36) we have the inverted form of (37):

(38) $$\tau^*(n) = \sum (-1)^i\ b(a_1)\ b(a_2) \ldots b(a_i) ,$$

where the sum extends over all compositions of n . The sequence $\{b(n)\}$ has in part the form $\{1,1,3,4,7,14,\ldots\}$.

The number of compositions of n in which no more than one part of each pair of consecutive parts is 1 can be found by setting $g(n) = 1$ and

(39) $$f(m,n) = \begin{cases} 0 , & \text{if } m = n = 1 \\ 1 , & \text{otherwise} \end{cases} .$$

Using $G(x) = x/(1-x)$ and $F(x,y) = (x+y-xy)/(1-x)(1-y)$ in (1.17), we have

$$(40) \qquad B(x,y) = \frac{xy}{1-xy}\,\{1+B(x,1)\} - xy\,\partial B(x,0) \quad ,$$

where $\partial B(x,0)$ denotes $\partial B(x,s)/\partial s$ at $s = 0$. Differentiating with respect to y at $y = 0$ in (40), or putting $y = 1$ in (40) gives two equations linear in $B(x,1)$ and $\partial B(x,0)$; solving for $B(x,1)$ gives

$$(41) \qquad B(x,1) = \frac{x(1+x^2)}{1-x-x^2-x^3} = x+x^2+3x^3+5x^4+9x^5+ \ldots \quad .$$

It is interesting to note the similarity between this generating function and the generating function given by taking $s = 1$ in (4.13) which generates the number of two row n-ominoes.

Now we are going to show how the common identities involving the generating functions of various partitions of n can be made to follow from the formulas of Chapter I. It has already been observed in Chapter IV that another parameter can be introduced which keeps track of the contribution to the sum which defines $b(n)$ made by compositions of n with exactly i parts. To do this, we simply write $zF(x,y)$ and $zG(x)$ in place of $F(x,y)$ and $G(x)$; then, for example, (1.15) becomes

$$(42) \qquad B_{k+1}(x,y,z) = \frac{z}{2\pi i}\int_c F(xy,\tfrac{1}{s})\, B_k(x,s,z)\,\frac{ds}{s} \quad ,$$

and hence,

$$B(x,y,z) = B_1(x,y)z + B_2(x,y)z^2 + \dots \quad , \tag{43}$$

where $B_k(x,y)$ is defined in (1.15). Also, it follows from (43) that

$$B(x,y,z) = zG(xy) + \frac{z}{2\pi i}\int_c F(xy,\frac{1}{s})\, B(x,y,z)\,\frac{ds}{s} \quad . \tag{44}$$

We note for purposes of comparison that the following technique gives a generating function in a certain form for most kinds of partition functions. Suppose $0 < a_1 < a_2 < \dots$ is a given set of integers and let $\{\alpha_{i1},\alpha_{i2},\dots\}$ be a set of non-negative integers associated with a_i, for $i = 1,2,\dots$. The number of partitions of n involving exactly k of the parts $a_1,a_2,\dots,$ where a_i is selected as a part exactly α_{ij} times, for some j and $i = 1,2,\dots$, is the coefficient of $z^k x^n$ in the expansion of

$$\Big\{\sum_{\nu} z^{\alpha_{1\nu}}\, x^{a_1\alpha_{1\nu}}\Big\}\Big\{\sum_{\nu} z^{\alpha_{2\nu}}\, x^{a_2\alpha_{2\nu}}\Big\}\dots \quad . \tag{45}$$

For example, if $\{\alpha_{i1},\alpha_{i2},\dots\} = \{0,1,\dots\}$, for $i = 1,2,\dots,$ the generating function is

$$\prod_{i=1}^{\infty}\{1+zx^{a_i}+z^2x^{2a_i}+\dots\} = \prod_{i=1}^{\infty}\frac{1}{(1-zx^{a_i})} \quad ; \tag{46}$$

also, if $\{\alpha_{i1},\alpha_{i2},\dots\} = \{0,1\}$, for $i = 1,2,\dots$, the generating function is

$$\prod_{i=1}^{\infty} (1+zx^{a_i}) \quad . \tag{47}$$

It follows from (46) that $p_k(n)$, the number of partitions of n into exactly k positive parts, is generated by

$$P(x,z) = \prod_{i=1}^{\infty} \frac{1}{(1-zx^i)} = 1 + \sum_{n=1}^{\infty} \sum_{k=1}^{n} p_k(n) z^k x^n \quad . \tag{48}$$

If $g(n) = 1$, and

$$f(m,n) = \begin{cases} 1 , & \text{if } m \leq n \\ 0 , & \text{otherwise} \end{cases} \quad , \tag{49}$$

then $b(a,n)$ is the number of partitions of n with smallest part equal to a . Alternatively, if $g(n) = 1$, and

$$f(m,n) = \begin{cases} 1 , & \text{if } m \geq n \\ 0 , & \text{otherwise} \end{cases} \quad , \tag{50}$$

then $b(a,n)$ is the number of partitions of n with largest part equal to a .

Using the definition of $\{f(m,n)\}$ given in (49), we have $F(x,y) = xy/(1-y)(1-xy)$; substituting this into (1.15) we obtain a recurrence satisfied by $\{B_k(x,y)\}$ which along with $B_1(x,y) = xy/(1-xy)$ implies

$$B_k(x,y) = \frac{x^k y}{(1-x) \ldots (1-x^{k-1})(1-x^k y)} \quad ; \tag{51}$$

hence,

(52) $$B(x,y,z) = \sum_{k=1}^{\infty} \frac{yx^k z^k}{(1-x) \dots (1-x^{k-1})(1-x^k y)} .$$

Similarly, the definition of $\{f(m,n)\}$ given in (50) implies $F(x,y) = xy/(1-x)(1-xy)$, and this leads to

(53) $$B_k(x,y) = \frac{x^k y}{(1-xy)(1-x^2y) \dots (1-x^k y)} ;$$

hence,

(54) $$B(x,y,z) = \sum_{k=1}^{\infty} \frac{yx^k z^k}{(1-xy)(1-x^2y) \dots (1-x^k y)} .$$

When $y = 1$ in (52) or (54) the generating functions are both equal to $P(x,z) - 1$; hence we obtain Euler's famous identity

(55) $$\prod_{i=1}^{\infty} \frac{1}{(1-zx^i)} = 1 + \sum_{k=1}^{\infty} \frac{x^k z^k}{(1-x)(1-x^2) \dots (1-x^k)} .$$

If we want the generating function for the number of partitions of n with the smallest part restricted in some way, we modify (51) and (52) by redefining $G(x)$. The generating functions (53) and (54) can be modified similarly to treat restrictions on the largest part in the partition.

Of course, $\{f(m,n)\}$ and $\{g(n)\}$ may be defined in many

ways so that $b(n)$ enumerates a variety of partitions of n. Using the technique of Chapter I, it generally develops that the functions $B^j(x,y,z)$ tend to a generating function having a form given by (45), while the function $B(x,y,z) \equiv B_1(x,y)z + B_2(x,y)z^2 + \ldots$ will usually have a form similar to (52) or (54). Equating these results sometimes leads to a classical identity due to Euler, Cauchy, Jacobi, et al.; generally, the methods used in the first place in establishing these identities are simpler and more elementary than an application of the methods of Chapter I. We have only mentioned this application to indicate a connection between the generating functions of partition functions and the Fredholm equation which is the connecting link between all of the problems discussed so far.

CHAPTER VII

GENERALIZED GRAPHS

Let $P_k(X)$ denote the collection of ordered k-subsets and $P^k(X)$ the collection of ordered k-selections of the set X. Let G be a permutation group of degree k whose elements permute the order of the terms in any k-tuple. Two k-tuples are G-equivalent if there is a permutation in G which transforms one into the other; clearly, G partitions $P_k(X)$ and $P^k(X)$ into collections of equivalence classes which we denote by $GP_k(X)$ and $GP^k(X)$ respectively. For example, if $G = S_k$, the symmetric group on k letters, $S_kP_k(X)$ denotes the collection of all unordered k-subsets of X since two k-tuples are the same if and only if they contain the same elements.

Now let G be a given subgroup of S_k and let $N_m = \{1,2,\ldots,m\}$; we call the elements of N_m nodes and the elements of $GP_k(N_m)$ edges. We write $G_k(m)$ for the set of all (k,G)-graphs on m nodes where a (k,G)-graph consists of the set of nodes N_m and some subset E of $GP_k(N_m)$ called the edge set of the graph. The subset of $G_k(m)$ containing graphs with exactly n edges will be denoted by $G_k(m,n)$.

If π is a permutation of N_m, π induces a permutation π^* of $GP_k(N_m)$ defined by

$$(1) \qquad \pi^* : (a_1,\ldots,a_k) \to (\pi(a_1),\ldots,\pi(a_k)) \, , \quad (a_1,\ldots,a_k) \in GP_k(N_m) \, .$$

Two graphs in $G_k(m)$ are <u>isomorphic</u> if there is a permutation of the nodes of one which transforms its edge set into that of the second graph. The set of non-isomorphic (k,G)-graphs with m nodes and n edges will be denoted by $G_k^*(m,n)$ and we define $G_k^*(m) = \bigcup_n G_k^*(m,n)$.

Let E denote the edge set of a graph $A \in G_k(m,n)$ and define $\overline{A} \in G_{m-k}(m,n)$ as the graph having the edge set $\overline{E} = \{N_m - x \mid x \in E\}$. This one to one correspondence shows that $|G_k(m,n)| = |G_{m-k}(m,n)|$; hence, $|G_k(m)| = |G_{m-k}(m)|$. Next, if $A, B \in G_k(m,n)$ are isomorphic, then $\overline{A}$ and $\overline{B}$ are isomorphic as well. To see this, suppose π is a permutation of N_m such that π^* transforms the edge set of A into that of B ; then $\pi^* x = y$, $x \in A$, $y \in B$ implies $\pi^*(N_m - x) = N_m - y$. This proves $|G_k^*(m,n)| = |G_{m-k}^*(m,n)|$ and from this it follows that $|G_k^*(m)| = |G_{m-k}^*(m)|$.

When $k = 2$, G may be chosen as either of two groups, the identity group I of degree 2 or the symmetric group S of degree 2 . The edges in $IP_2(m)$ and $SP_2(m)$ are said to be directed and undirected edges respectively; also, $I_2(m)$, $S_2(m)$, $I_2^*(m)$, $S_2^*(m)$ are called the labelled directed, and linear graphs on m nodes, and the non-isomorphic directed, and linear graphs on m nodes, respectively. We are going to consider the problem of enumerating the sets $I_k(m)$, $S_k(m)$, $I_k^*(m)$, and $S_k^*(m)$, where I and S denote the identity and symmetric groups of degree k , respectively. To do this we will require Pólya's enumeration theorem which we state here for convenience.

Suppose S is a set of distinct objects and that $w(n)$ objects have been assigned the weight n, for $n = 0,1,\ldots$; we define $f(x) = \sum w(n)x^n$. If G is a permutation group of degree k, then the weight of an equivalence class in $GP^k(S)$ is the sum of the weights of the objects contained in a representitive k-tuple in the equivalence class. Letting $W(n)$ denote the number of equivalence classes with weight n, for $n = 0,1,\ldots$, we define $F(x) = \sum W(n)x^n$. Pólya's theorem states that

$$(2) \qquad F(x) = Z(G;f(x), f(x^2), \ldots, f(x^k)) \quad ,$$

where

$$(3) \qquad Z(G;f_1,f_2,\ldots,f_k) = \frac{1}{|G|} \sum_{\pi \in G} c(\pi) \quad ,$$

and $c(\pi) = f_1^{j_1} f_2^{j_2} \ldots f_k^{j_k}$ if π has j_i cycles of length i for $i = 1,2,\ldots,k$; $Z(G;f_1,\ldots,f_k)$ is the <u>cycle index</u> of G .

Before proceeding, we note that if the number of edges in $GP_k(N_m)$ is known, then it is easy to find the number of graphs in $G_k(m)$ and $G_k(m,n)$. In fact, if $|GP_k(N_m)| = N$, we have $|G_k(m,n)| = \binom{N}{n}$ and $|G_k(m)| = 2^N$. Thus, since $|IP_k(N_m)| = k!\binom{m}{k}$ and $|SP_k(N_m)| = \binom{m}{k}$ we have

$$(4) \qquad |I_k(m,n)| = \binom{k!\binom{m}{k}}{n} \; , \quad |I_k(m)| = 2^{k!\binom{m}{k}} \quad ,$$

$$(5) \qquad |S_k(m,n)| = \binom{\binom{m}{k}}{n}, \quad |S_k(m,n)| = 2^{\binom{m}{k}} .$$

For the time being we assume that G is the identity group I of degree k ; thus, $IP_k(N_m)$ denotes the set of ordered subsets of N_m . We want to use Pólya's theorem to determine the number of elements in $I_k^*(m,n)$ and $I_k^*(m)$. In the application of the theorem, the set S consists of the two objects "is not contained in the edge set" and "is contained in the edge set" which we weight 0 and 1 respectively. If the elements of $IP_k(N_m)$ are put in some fixed order, say $\left(X_1, X_2, \ldots, X_{k!\binom{m}{n}}\right)$, each element of $IP^{k!\binom{m}{k}}(S)$ may be interpreted as a description of a graph contained in $I_k(N_m)$ by saying the i^{th} component of $X \in IP^{k!\binom{m}{k}}(S)$ tells whether X_i is or is not contained in the edge set of the (k,I)-graph being described. The permutation group $\mathbb{G}$ of $IP^{k!\binom{m}{k}}(S)$ is induced by the symmetric group on N_m as described in (1). To apply the theorem, we need the cycle index of G and use $f(x) = 1 + x$.

Let π denote a permutation of N_m with j_i cycles of length i involving the ij_i nodes contained in J_i for $i = 1,2,\ldots,m$. We want to determine the number of cycles of length i in π^* for $i = 1,2,\ldots,k!\binom{m}{k}$. Note that π^* maps an edge having a_i nodes in J_i onto another edge having a_i nodes in J_i for $i = 1,2,\ldots,m$. It is easy to see that the edge cycles induced by π among the set of edges having a_i nodes in J_i with $a_1 + a_2 + \ldots + a_m = k$,

$a_i \geq 0$, have length $\lambda(a_i)$, the least common multiple of the subscripts of the non-zero a's . Furthermore, if the non-zero a's are $a_{i_1}, a_{i_2}, \ldots,$ then the number of edges of this kind is

$$k! \binom{i_1 j_{i_1}}{a_{i_1}} \binom{i_2 j_{i_2}}{a_{i_2}} \cdots = \nu(a_i) \quad . \tag{6}$$

This follows from the fact that there are $\binom{ij_i}{a_i}$ ways to select a_i elements from J_i for $i = i_1, i_2, \ldots$ and $k!$ ways to distribute these elements in a k-tuple. Thus, π induces $\nu(a_i)/\lambda(a_i)$ edge cycles of length $\lambda(a_i)$ in the set of edges having a_i nodes in J_i for $i = 1,2,\ldots,m$ in each edge. Letting $(a_1, a_2, \ldots, a_m)$ range over all compositions of k into non-negative parts, we obtain a complete list of the cycles of π^* .

If we write $c(\pi) = f_1^{j_i} f_2^{j_2} \ldots$ and $c(\pi^*) = g_1^{\ell_1} g_2^{\ell_2} \ldots$ explicit formulas can be obtained using the result of the last paragraph. For example, the contributions to $c(\pi^*)$ when $a_r = k$, or $a_r + a_s = k$, $a_r, a_s \neq 0$, or $a_r + a_s + a_t = k$, $a_r, a_s, a_t \neq 0$, are respectively

$$f_r^{j_r} \rightarrow g_r^{k! \binom{rj_r}{k} / r} \quad , \tag{7}$$

(8) $$f_r^{j_r} f_s^{j_s} \to g_{[r,s]}^{k!\sum \binom{rj_r}{a_r}\binom{sj_s}{a_s}/[r,s]} ,$$

(9) $$f_r^{j_r} f_s^{j_s} f_t^{j_t} \to g_{[r,s,t]}^{k!\sum \binom{rj_r}{a_r}\binom{sj_s}{a_s}\binom{tj_t}{a_t}/[r,s,t]} ,$$

where the sums in (8) and (9) extend over all compositions (a_r, a_s) and (a_r, a_s, a_t) of k into positive parts.

Using these formulas it is not difficult to find the cycle index involved when $m = 6$ and $k = 3$; it is

(10) $$\frac{1}{6!}\{g_1^{120} + 60g_2^{60} + 40g_3^{40} + 180g_4^{30} + 144g_5^{24} + 120g_6^{20} + 15g_1^{24}g_2^{48} + 40g_1^{6}g_3^{38} + 120g_2^{3}g_3^{8}g_6^{15}\} .$$

The number of non-isomorphic $(3, I)$-graphs with n edges and 6 nodes is the coefficient of x^n in (10) when $1 + x^i$ is substituted for g_i.

Now we assume $G = S$, the symmetric group of degree k; thus, elements of $SP_k(N_m)$ are simply k-subsets of N_m. It is easy to see that $|S_1^*(m,n)| = 1$ and $|S_1^*(m)| = m+1$. The generating series for $|S_2^*(m,n)|$, the number of linear graphs with n edges, was given by Pólya [16]; subsequent expositions of this result have been given by Harary [8] and Riordan [19]. The cycle index of the

permutation group induced in $SP_k(N_m)$ by the permutations of N_m seems to be complicated for general k ; the difficulties are illustrated below in the case $k = 3$.

Suppose a permutation π of N_m has j_i cycles of length i and let J_i denote the set containing the ij_i nodes involved in these cycles for $i = 1,2,\ldots,m$. We begin by determining the edge cycles induced in $SP_3(J_i)$ by π .

Case 1: If $i \equiv 0 \pmod 3$, the edge cycles of $SP_3(J_i)$ are of two kinds; first, there are edge cycles which involve only edges having the form $\{a, \pi^{i/3}(a), \pi^{2i/3}(a)\}$, $a \in J_i$, and second, edge cycles which involve only edges having the form $\{a,b,c\}$, $a,b,c \in J_i$, not of the first kind. There are $ij_i/3$ edges of the first kind and these edges split into cycles of length $i/3$, so there are j_i edge cycles of length $i/3$ induced by π among these edges. The remaining edges are $\binom{ij_i}{3} - ij_i/3$ in number and π induces edge cycles of length i among these edges, so there are $\frac{1}{i}\left\{\binom{ij_i}{3} - ij_i/3\right\}$ edge cycles of length i .

Case 2: If $i \equiv 1,2 \pmod 3$ the $\binom{ij_i}{3}$ edges in $SP_3(J_i)$ split into cycles of length i ; hence, there are $\frac{1}{i}\binom{ij_i}{3}$ edge cycles of length i induced in $SP_3(J_i)$ by π .

Next, we determine the edge cycles induced in $SP_3(J_{i_1} \cup J_{i_2}) - SP_3(J_{i_1}) - SP_3(J_{i_2})$ by π when $i_1 \neq i_2$; it is necessary to distinguish

three cases depending on the parity of i_1 and i_2.

Case 3: If i_1 and i_2 are both odd, then the edges having the form $\{a,b,c\}$ with $a \in J_{i_1}$, $b,c \in J_{i_2}$, are $i_1 j_{i_1} \cdot \binom{i_2 j_{i_2}}{2}$ in number, and they split into cycles of length $[i_1,i_2]$, the least common multiple of i_1 and i_2. Thus, there are $j_{i_1} j_{i_2} (i_1,i_2)(i_2 j_{i_2} - 1)/2$ edge cycles of this type; interchanging the subscripts 1 and 2 gives the number of remaining edge cycles also of length $[i_1,i_2]$. Altogether there are $j_{i_1} j_{i_2} (i_1,i_2)(i_1 j_{i_1} + i_2 j_{i_2} - 2)/2$ edge cycles of length $[i_1,i_2]$ induced by π among the edges of $SP_3(J_{i_1} \cup J_{i_2}) - SP_3(J_{i_1}) - SP_3(J_{i_2})$ when i_1 and i_2 are both odd.

Case 4: If i_1 and i_2 are of opposite parity, say i_2 is even, the edge cycles induced by π are of three kinds; first, those involving edges having the form $\{a,b,\pi^{i_2/2}(b)\}$ with $a \in J_{i_1}$, $b \in J_{i_2}$; second, those involving the edges having the form $\{a,b,c\}$ with $a \in J_{i_1}$, $b,c, \in J_{i_2}$, $c \neq \pi^{i_2/2}(b)$; third, those involving edges having the form $\{a,b,c\}$ with $a,b \in J_{i_1}$, $c \in J_{i_2}$. There are $i_1 j_{i_1} \cdot i_2 j_{i_2}/2$ edges of the first form and these split into cycles of length $[i_1, i_2/2]$; hence, there are $j_{i_1} j_{i_2} \cdot (i_1, i_2/2)$ edge cycles of length $[i_1, i_2/2]$ induced in this set. The edges of the second form are $i_1 j_{i_1} \, i_2 j_{i_2} (i_2 j_{i_2} - 2)/2$ in number and they form cycles of length $[i_1,i_2]$; hence, there are

$j_{i_1} j_{i_2} (i_1,i_2)(i_2 j_{i_2} -2)/2$ cycles of length $[i_1,i_2]$ induced in this set. Finally, the $i_2 j_{i_2} \cdot \binom{i_1 j_{i_1}}{2}$ edges having the third form split into cycles of length $[i_1,i_2]$ as well, so there are $j_{i_1} j_{i_2} (i_1,i_2)(i_1 j_{i_1} -2)/2$ cycles of this length induced among the edges in the last set.

Summing up these results, we have that if i_1 is odd and i_2 is even, then there are $j_{i_1} j_{i_2} (i_1,i_2)/2$ edge cycles of length $[i_1,i_2/2]$ and $j_{i_1} j_{i_2} (i_1,i_2)(i_1 j_{i_1} + i_2 j_{i_2} -3)/2$ cycles of length $[i_1,i_2]$ induced in the set $SP_3(J_{i_1} \cup J_{i_2}) - SP_3(J_{i_1}) - SP_3(J_{i_2})$ by π .

Case 5: If i_1 and i_2 are both even, we can ignore the parity of i_1 for the moment and determine the edge cycles among the edges having the forms $\{a,b,\pi^{i_2/2}(b)\}$, $a \in J_{i_1}$, $b \in J_{i_2}$, and $\{a,b,c\}$, $a \in J_{i_1}$, $b,c, \in J_{i_2}$, $c \neq \pi^{i_2/2}(b)$, just as in Case 4 above. This gives $2j_{i_1} j_{i_2} (i_1,i_2/2)$ edge cycles of length $[i_1,i_2/2]$ and $j_{i_1} j_{i_2} (i_1,i_2)(i_2,j_{i_2} -2)/2$ edge cycles of length $[i_1,i_2]$. By interchanging the subscripts 1 and 2 we obtain the number of remaining edge cycles. In conclusion, if i_1 and i_2 are both even, then there are $j_{i_1} j_{i_2} (i_1,i_2/2)$ cycles of length $[i_1,i_2/2]$, $j_{i_1} j_{i_2} (i_1/2,i_2)$ cycles of length $[i_1/2,i_2]$, and $j_{i_1} j_{i_2} (i_1,i_2)(i_1 j_{i_1} + i_2 j_{i_2} -4)/2$ cycles of length $[i_1,i_2]$ induced by π among the edges of $SP_3(J_{i_1} \cup J_{i_2}) - SP_3(J_{i_1}) - SP_3(J_{i_2})$.

Finally, there are edge cycles induced by π involving nodes chosen from three different sets J_i .

Case 6: There are $i_1 j_{i_1} i_2 j_{i_2} i_3 j_{i_3}$ edges having the form $\{a,b,c\}$ with $a \in J_{i_1}$, $b \in J_{i_2}$, $c \in J_{i_3}$ and π induces $i_1 j_{i_1} i_2 j_{i_2} i_3 j_{i_3} / [i_1, i_2, i_3]$ cycles of length $[i_1, i_2, i_3]$ among these edges.

If we write $c(\pi) = f_1^{i_1} f_2^{i_2} \ldots$ and $c(\pi^*) = g_1^{\ell_1} g_2^{\ell_2} \ldots$, where π is a permutation of N_m and π^* is the permutation induced by π in $SP_3(N_m)$, then we have the following correspondence between the factors of $c(\pi)$ and $c(\pi^*)$:

$$(11)\qquad f_i^{j_i} \to \begin{cases} g_{i/3}^{j_i}\, g_i^{[\binom{ij_i}{3} - ij_i/3]/3}, & \text{if } i \equiv 0 \pmod 3 \\[2ex] g_i^{\binom{ij_i}{3}/i}, & \text{if } i \equiv 1,2 \pmod 3 \end{cases}$$

$$(12)\qquad f_{i_1}^{j_{i_1}}\, f_{i_2}^{j_{i_2}} \to \begin{cases} g_{[i_1,i_2]}^{j_{i_1}j_{i_2}(i_1,i_2)(i_1j_{i_1}+i_2j_{i_2}-2)/2}, & \text{if } i_1,i_2 \equiv 1 \pmod 2 \\[2ex] g_{[i_1,i_2/2]}^{j_{i_1}j_{i_2}(i_1,i_2/2)}\quad g_{[i_1,i_2]}^{j_{i_1}j_{i_2}(i_1,i_2)(i_1j_{i_1}+i_2j_{i_2}-3)/2}, & \text{if } i_1 \equiv 1 \pmod 2 \text{ and } \\ & i_2 \equiv 0 \pmod 2 \\[2ex] g_{[i_1,i_2/2]}^{j_{i_1}j_{i_2}(i_1,i_2/2)}\quad g_{[i_1/2,i_2]}^{j_{i_1}j_{i_2}(i_1/2,i_2)}\quad g_{[i_1,i_2]}^{j_{i_1}j_{i_2}(i_1,i_2)(i_1j_{i_1}+i_2j_{i_2}-4)/2}, & \\ & \text{if } i_1,i_2 \equiv 0 \pmod 2 \end{cases}$$

$$(13)\qquad f_{i_1}^{j_{i_1}}\, f_{i_2}^{j_{i_2}}\, f_{i_3}^{j_{i_3}} \to g_{[i_1,i_2,i_3]}^{i_1j_{i_1}i_2j_{i_2}i_3j_{i_3}/[i_1,i_2,i_3]}$$

When applying these formulas, the pair or triple of subscripts i_1, i_2 or i_1, i_2, i_3 is considered exactly once. Using Riordan's [19] list of cycle indices for the symmetric groups on N_m and the correspondences given in (11), (12), and (13), we get the following cycle indices for the groups induced in $SP_3(N_m)$ for $m = 6, 7, 8,$ and 9, respectively.

(14) $$\frac{1}{6!}\{g_1^{20} + 15g_1^8g_2^6 + 45g_1^4g_2^8 + 80g_1^2g_3^6 + 120g_1^2g_3^2g_6^2 + 15g_2^{10} + 180g_2^2g_4^4 + 120g_2g_6^3 + 144g_5^4\}$$

(15) $$\frac{1}{7!}\{g_1^{35} + 21g_1^{15}g_2^{10} + 105g_1^7g_2^{14} + 70g_1^5g_3^{10} + 420g_1^4g_2g_3^4g_6^3 + 105g_1^3g_2^{16} + 280g_1^2g_3^{11} + 840g_1g_2^3g_4^7 + 210g_1g_2^2g_3^2g_6^4 + 420g_1g_4g_6g_{12}^2 + 840g_2g_3g_6^5 + 504g_5^7 + 504g_5^3g_{10}^2 + 720g_7^5\}$$

(16) $$\frac{1}{8!}\{g_1^{56} + 28g_1^{26}g_2^{15} + 210g_1^{12}g_2^{22} + 112g_1^{11}g_3^{15} + 420g_1^6g_2^{25} + 1120g_1^5g_2^3g_3^7g_6^4 + 420g_1^4g_2^4g_4^{11} + 1680g_1^3g_2^4g_3^3g_6^6 + 1120g_1^2g_3^8g_6^5 + 1120g_1^2g_3^{18} + 2520g_1^2g_2^5g_4^{11}$$

$$+ 3360g_1g_2g_3g_4^2g_6g_{12}^3 + 2688g_1g_5^2g_{15}^3 + 4032g_1g_5^5g_{10}^3$$

$$+ 1344g_1g_5^{11} + 105g_2^{28} + 1260g_2^6g_4^{11} + 3360g_2g_6^9$$

$$+ 3360g_2g_3^2g_6^8 + 1260g_4^{14} + 5760g_7^8 + 5040g_8^7 \}$$

(17) $$\frac{1}{9!} \{g_1^{84} + 36g_1^{42}g_2^{21} + 378g_1^{20}g_2^{32} + 168g_1^{21}g_3^{21} + 1260g_1^{10}g_2^{37}$$

$$+ 2520g_1^9g_2^6g_3^{11}g_6^5 + 756g_1^{10}g_4^{16}g_2^5 + 945g_1^4g_2^{40}$$

$$+ 7560g_1^5g_2^8g_3^5g_6^8 + 3360g_1^3g_3^{27} + 7560g_1^4g_2^8g_4^{16}$$

$$+ 3024g_1^4g_5^{16} + 2520g_1g_2^{10}g_3^3g_6^9 + 10080g_1^3g_3^{13}g_6^7$$

$$+ 11340g_1^2g_2^9g_4^{16} + 15120g_1g_2^2g_3^3g_4^4g_6g_{12}^4 + 18144g_1^2g_2g_5^8g_{10}^4$$

$$+ 10080g_1g_2g_3^3g_6^{12} + 2240g_1^3g_3^{27} + 15120g_1g_2^2g_3g_4^4g_6^2g_{12}^4$$

$$+ 9072g_2^2g_5^4g_{10}^6 + 11340g_2^2g_4^{20} + 24192g_1g_3g_5^4g_{15}^4$$

$$+ 30240g_1g_2g_3g_6^{13} + 25920g_7^{12} + 18144g_4g_5^2g_{10}g_{20}^3$$

$$+ 25920g_7^6g_{14}^3 + 45360g_4g_8^{10} + 40320g_9^{12} \}$$

The number of non-isomorphic (3,S)-graphs with n edges and m nodes, for $m = 6,7,8,9,$ and $n = 0,1,\ldots,m$, is the coefficient of x^n in (14), (15), (16), and (17) respectively, when $1 + x^i$ is substituted for g_i in each of these relations.

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